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ABSTRACT

This is one in a series of SMSG supplementary and enrichment pamphlets for high school students. This series is designed to make material for the study of topics of special interest to students readily accessible in classroom quantity. Topics covered include directed segments, applications, components, and inner products. (MP)

SCHOOL
MATHEMATICS
STUDY GROUP

SP-6

SUPPLEMENTARY and ENRICHMENT SERIES

THE SYSTEM OF VECTORS

Edited by Karl Kalman

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PREFACE

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which, though within the grasp of secondary school students, do not find a place in the curriculum simply because of a lack of time.

Many classes and individual students, however, may find time to pursue mathematical topics of special interest to them. This series of pamphlets, whose production is sponsored by the School Mathematics Study Group, is designed to make material for such study readily accessible in classroom quantity.

Some of the pamphlets deal with material found in the regular curriculum but in a more extensive or intensive manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum. It is hoped that these pamphlets will find use in classrooms in at least two ways. Some of the pamphlets produced could be used to extend the work done by a class with a regular textbook but others could be used profitably when teachers want to experiment with a treatment of a topic different from the treatment in the regular text of the class. In all cases, the pamphlets are designed to promote the enjoyment of studying mathematics.

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THE SYSTEM OF VECTORS

Introduction: To the student

We hope you will find the material in this pamphlet interesting, stimulating and rewarding. You will experience little difficulty in understanding the concept of a vector even though your preparation in Mathematics may not be very extensive but at the same time a reasonable mastery of the subject will prepare you for more advanced work. Of the many new ideas you will encounter in these pages we should like to mention three which are of particular interest. First, we hope you will be impressed with the solution of Geometry problems by vector methods. Also, this material has tremendous application to problems in Physics and this is demonstrated in the text, although you should be reminded that this section requires a knowledge of some elementary Trigonometry. Finally, should try yourself out on the last section which is concerned with Vectors as a Formal System.

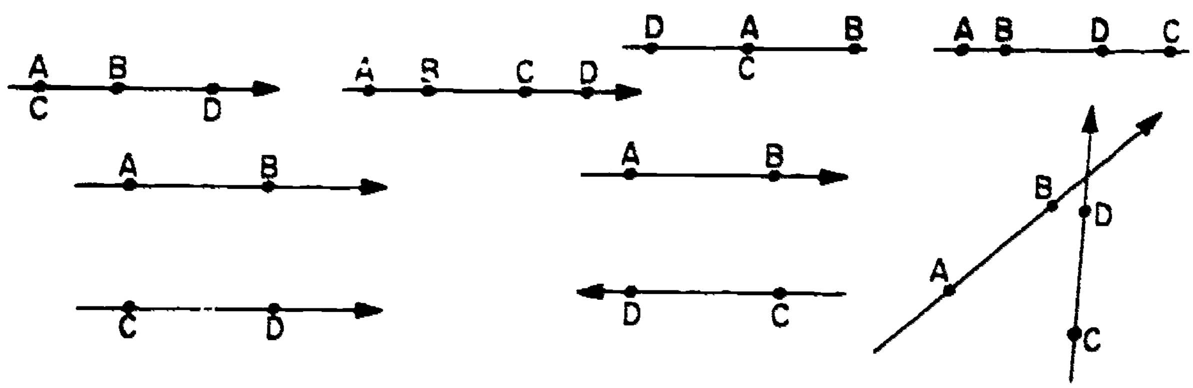
CONTENTS

THE SYSTEM OF VECTORS

1. Directed Line Segments.

It is assumed in this pamphlet that you are familiar with plane geometry. We review some of the symbols of geometry. $\overleftrightarrow{AB}$ means the line which contains the distinct points A and B . $\overrightarrow{AB}$ means the ray whose vertex is A and which also contains the point B . $|AB|$ means the distance from A to B (and from B to A). It is a positive real number if A and B are distinct. It is zero if A and B are the same.

We need one further idea which is not ordinarily covered in geometry--that of parallel rays. Rays are said to be <u>parallel</u> if they lie on lines which are either parallel or coincident, and if they are similarly sensed. Figure 1a shows typical instances of rays which are parallel and of rays which are not parallel, and is supposed to take the place of a formal definition.



rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are Parallel

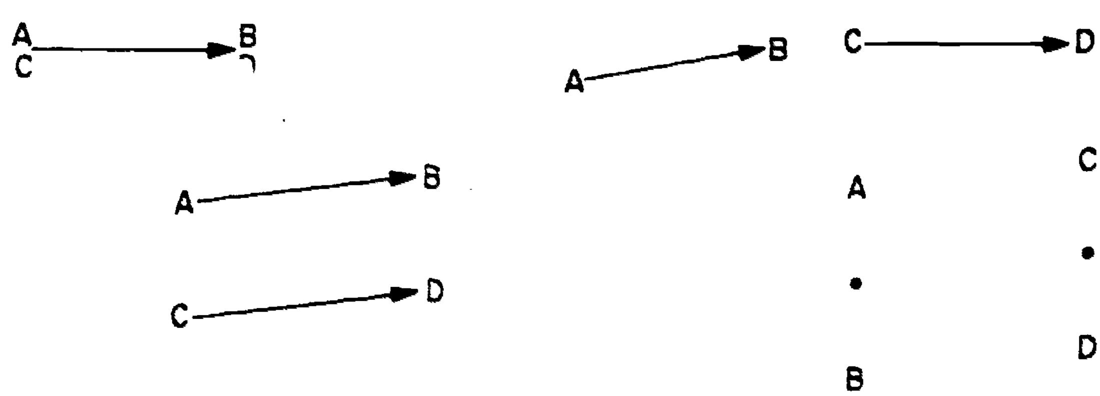
rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are not Parallel

Figure 1a

DEFINITION 1a. A line segment is said to be a directed line segment if one of its endpoints is designated as its initial point and the other endpoint is designated as its terminal point. We use the symbol $\overrightarrow{AB}$ to denote the directed line segment whose initial point is A and whose terminal point is B .

We say that directed line segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equivalent if it is true that their lengths are the same and also that the rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel. We write $\overrightarrow{AB} \doteq \overrightarrow{CD}$ to denote the fact that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equivalent.

Note: We consider that a single point can be both initial and terminal point of the same directed line segment and we consider that all such directed line segments are equivalent to one another.



$\overrightarrow{AB} \doteq \overrightarrow{CD}$

Figure 1b

Figure 1b shows some pairs of equivalent directed line segments. It uses the convention that the endpoint of a segment which has an arrow is the terminal point of the segment. Notice that if A, B, C, D are not collinear, then $\overrightarrow{AB} \doteq \overrightarrow{CD}$ if and only if ABDC is a parallelogram. We need the fact that if $\overrightarrow{AB}$ is any directed line segment and if C is any point, then there is one and only one point D such that $\overrightarrow{AB} \doteq \overrightarrow{CD}$. We do not prove this fact, but assume that it is known from the study of geometry.

<u>DEFINITION 1b</u>. Let $\overrightarrow{AB}$ and $\overrightarrow{CD}$ be any two directed line segments. Then by their <u>sum</u> $\overrightarrow{AB} + \overrightarrow{CD}$ we mean the directed line segment $\overrightarrow{AX}$, where X is the unique point such that $\overrightarrow{BX} \doteq \overrightarrow{CD}$.

We call the operation which assigns their sum to each pair of directed line segments the <u>addition</u> operation for directed line segments.

Figure 1c shows some sums of directed line segments.

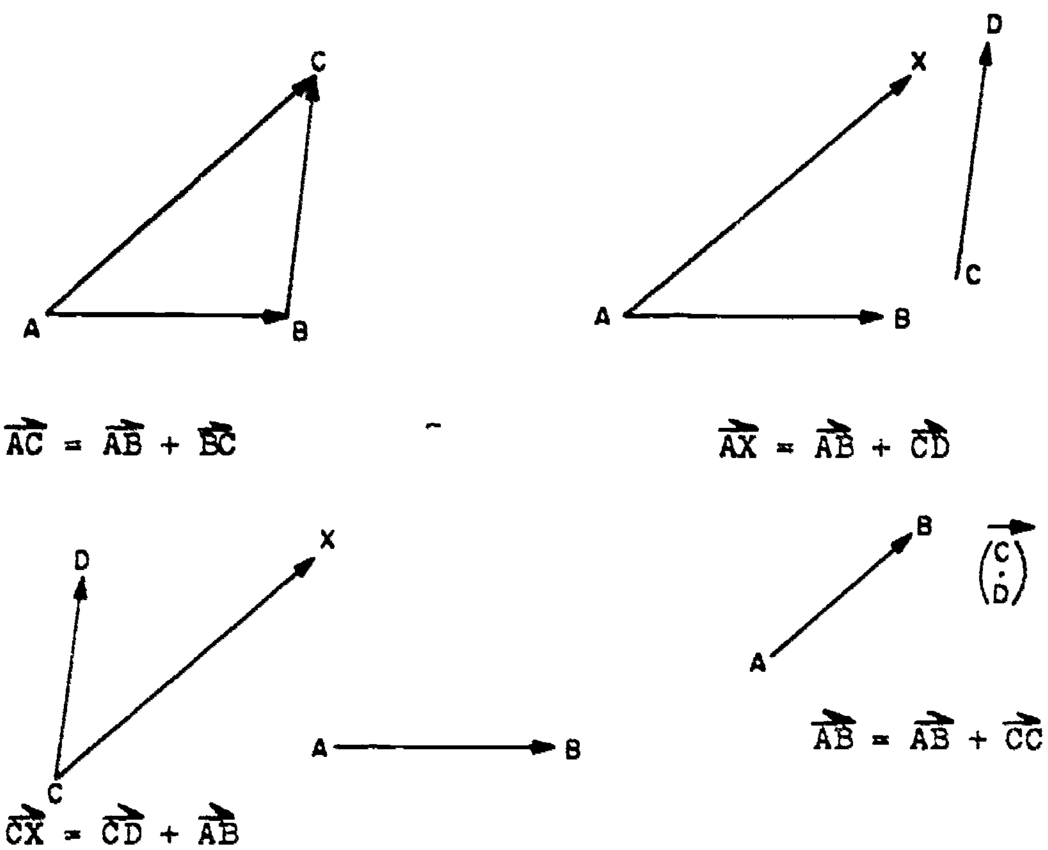


Figure 1c

Directed line segments can be added and multiplied by real numbers in a useful way. We give the formal definition of these operations here. Their properties are studied and applied throughout the rest of the chapter.

<u>DEFINITION</u>. Let $\overrightarrow{AB}$ be any directed line segment and let r be any real number. Then the product $r\overrightarrow{AB}$ is the directed line segment $\overrightarrow{AX}$, where X is determined as follows:

(1) If $r > 0$, then X is on the ray $\overrightarrow{AB}$ and $|AX| = r|AB|$.

(2) If $r < 0$, then X is on the ray opposite to $\overrightarrow{AB}$ and $|AX| = -r|AB|$.

(3) If $r = 0$, then $X = A$.

(4) If $B = A$, then $X = A$.

Figure 1d shows some typical products.

$$0\ \overrightarrow{AB} = \overrightarrow{AA}$$

$$1\ \overrightarrow{AB} = \overrightarrow{AB}$$

$$2\ \overrightarrow{AB} = \overrightarrow{AC}$$

$$\tfrac{1}{2}\ \overrightarrow{AB} = \overrightarrow{AD}$$

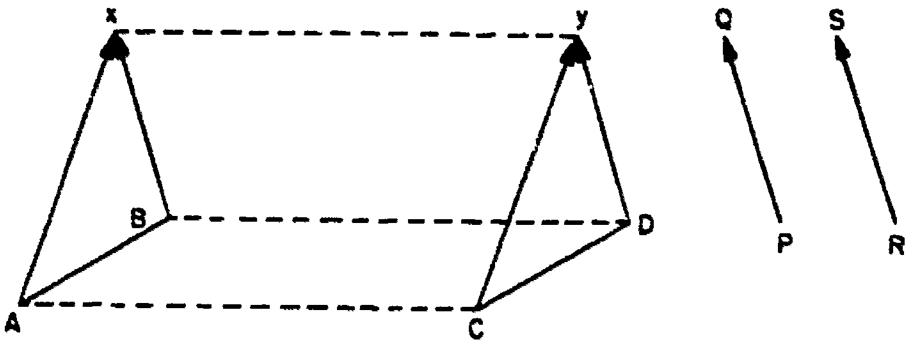
$$-1\ \overrightarrow{AB} = \overrightarrow{AE}$$

$$-2\ \overrightarrow{AB} = \overrightarrow{AF}$$

Figure 1d

It is useful to know that if equivalent directed line segments are added to equivalent directed line segments the sums are equivalent, and that if equivalent directed line segments are multiplied by the same number the products are equivalent. We now state these facts formally as theorems and illustrate them.

<u>THEOREM 1a</u>. If $\overrightarrow{AB} \doteq \overrightarrow{CD}$ and if $\overrightarrow{PQ} \doteq \overrightarrow{RS}$ then $\overrightarrow{AB} + \overrightarrow{PQ} \doteq \overrightarrow{CD} + \overrightarrow{RS}$.

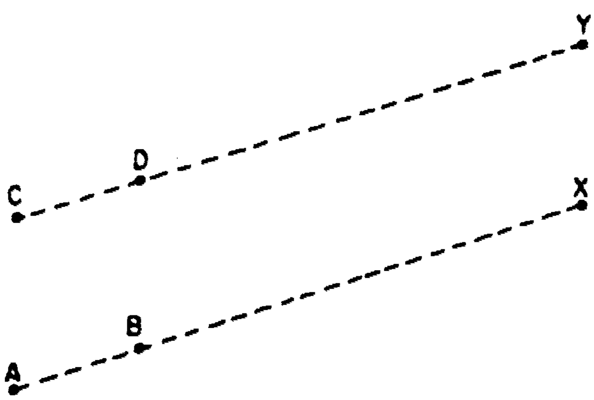


$\overrightarrow{AX} = \overrightarrow{AB} + \overrightarrow{PQ}$ $\qquad$ $\overrightarrow{AX} \doteq \overrightarrow{CY}$ $\qquad$ $\overrightarrow{CY} = \overrightarrow{CD} + \overrightarrow{PS}$

Figure 1e

Figure 1e shows a typical instance of this theorem. It is equivalent to the fact that if ABCD is a parallelogram and if XYDB is a parallelogram, then AXYC is a parallelogram. This is a special case of a famous theorem of geometry known as Desargues' Theorem.

<u>THEOREM 1b</u>. If $\overrightarrow{AB} \doteq \overrightarrow{CD}$ and if r is any real number, then $r\overrightarrow{AB} \doteq r\overrightarrow{CD}$.



$\overrightarrow{AX} = r\overrightarrow{AB}$; $\quad \overrightarrow{CY} = r\overrightarrow{CD}$; $\quad \overrightarrow{AX} \doteq \overrightarrow{CY}$.

Figure 1f

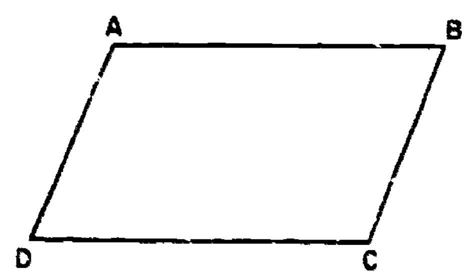
Figure 1f illustrates a case in which A, B, C, D are not collinear. It also illustrates the geometric version of the statement, that if ABDC is a parallelogram and if AX = CY then AXYC is a parallelogram.

Exercises 1

1. A and B are distinct points. List all the directed line segments they determine.

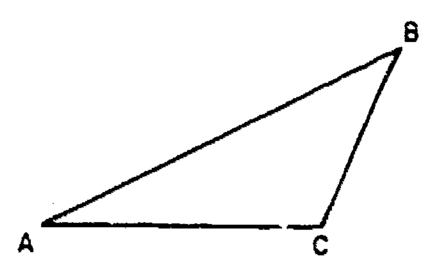
2. A, B and C are distinct points. List all the directed line segments they determine.

3. A, B, C and D are vertices of a parallelogram. List all the directed line segments they determine, and indicate which pairs are equivalent.



4. In triangle ABC

(a) $\overrightarrow{AB} + \overrightarrow{BC} = ?$
(b) $\overrightarrow{BA} + ? = \overrightarrow{BC}$.
(c) $? + \overrightarrow{BA} = \overrightarrow{BC}$.
(d) $? + \overrightarrow{AB} = \overrightarrow{AA}$.
(e) $(\overrightarrow{AB} + \overrightarrow{BC}) - \overrightarrow{CA} = ?$
(f) $\overrightarrow{BA} + (\overrightarrow{AC} + \overrightarrow{CB}) = ?$
(g) $? + \overrightarrow{AC} = \overrightarrow{CB}$.



5. A, B and X are collinear points. Find r such that

$$\overrightarrow{AX} = r\overrightarrow{AB}$$

and s such that

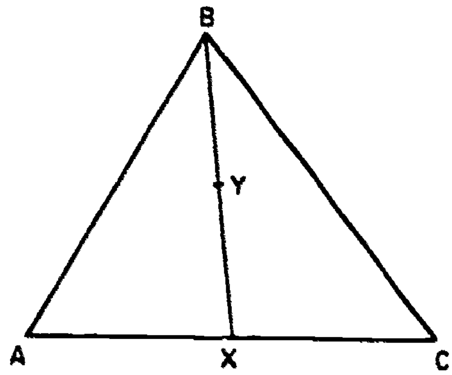
$$\overrightarrow{BX} = s\overrightarrow{BA} ,$$

if

(a) X is the midpoint of segment $\overline{AB}$.
(b) B is the midpoint of segment $\overline{AX}$.
(c) A is the midpoint of segment $\overline{BX}$.
(d) X is two-thirds of the way from A to B .
(e) B is two-thirds of the way from A to X .
(f) A is two-thirds of the way from B to X .

6. In triangle ABC , X is the midpoint of $\overline{AC}$ and Y is the midpoint of segment $\overline{BX}$.

(a) $\overrightarrow{BX} = \overrightarrow{BA} + ?\overrightarrow{AC}$.
(b) $\overrightarrow{BX} = ?\overrightarrow{BY}$.
(c) $\overrightarrow{BX} = \overrightarrow{BC} + ?$
(d) $\overrightarrow{BX} = \overrightarrow{BC} + \frac{1}{2}\ ?$
(e) $\overrightarrow{BY} = ?\overrightarrow{BX}$.
(f) $\overrightarrow{BY} = ?(\overrightarrow{BA} + \overrightarrow{AX})$.
(g) $\overrightarrow{BC} = ?\overrightarrow{BY} + \overrightarrow{XC}$.



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2. Applications to Geometry.

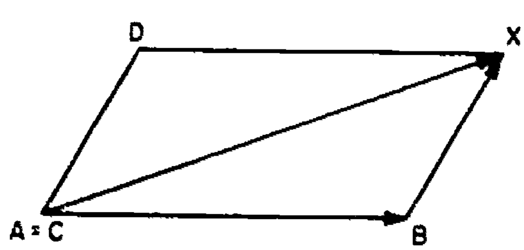
It is possible to use directed line segments to prove theorems of geometry. These proofs are based on algebraic properties of directed line segments. They are quite different from proofs usually given in geometry which appeal to such matters as congruent triangles and the like.

We state and illustrate the necessary algebraic properties of directed line segments here. We prove these statements in Section 3.

I. Commutative Law:

$$\overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{CD} + \overrightarrow{AB} .$$

Figure 2a shows an instance of the commutative law for addition in which the directed line segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have a common initial point.



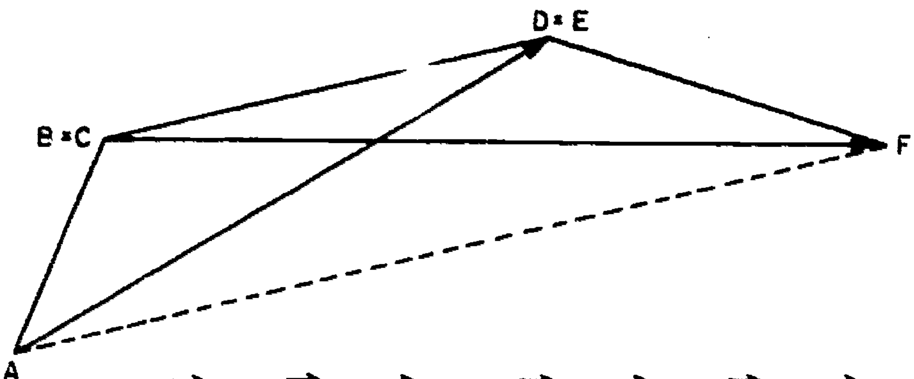
$$\overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{CD} + \overrightarrow{AB} = \overrightarrow{AX}$$

Figure 2a

II. Associative Law:

$$\overrightarrow{AB} + (\overrightarrow{CD} + \overrightarrow{EF}) = (\overrightarrow{AB} + \overrightarrow{CD}) + \overrightarrow{EF} .$$

Figure 2b shows sums $\overrightarrow{AB} + (\overrightarrow{CD} + \overrightarrow{EF})$ in which B and C are the same and D and E are the same.



$$\overrightarrow{AB} + (\overrightarrow{CD} + \overrightarrow{EF}) = (\overrightarrow{AB} + \overrightarrow{CD}) + \overrightarrow{EF} = \overrightarrow{AF}$$

Figure 2b

III. Existence of Zero Elements.

Every directed line segment of the type $\overrightarrow{AA}$ is a zero element because $\overrightarrow{PQ} + \overrightarrow{AA} = \overrightarrow{PQ}$.

IV. Existence of Additive Inverses.

$\overrightarrow{BA}$ is the additive inverse of $\overrightarrow{AB}$, because $\overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{AA}$.

We use a negative sign to denote the additive inverse of a directed line segment $\overrightarrow{AB}$, and write $-\overrightarrow{AB}$ for $\overrightarrow{BA}$. We also write $\overrightarrow{PQ} - \overrightarrow{AB}$ for $\overrightarrow{PQ} + \overrightarrow{BA}$.

This operation of subtraction is illustrated in Figure 2c.

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

$$\overrightarrow{AC} - \overrightarrow{AB} = \overrightarrow{BC}$$

$$\overrightarrow{AC} + \overrightarrow{BA} = \overrightarrow{BC}$$

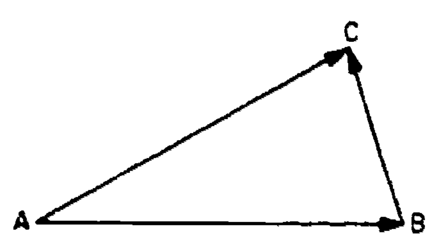


Figure 2c

V. The Associative Law.

$$r(s\overrightarrow{AB}) = (rs)\overrightarrow{AB} .$$

Figure 2d

$$-\tfrac{1}{2}(4\overrightarrow{AB}) = -\tfrac{1}{2}(\overrightarrow{AC}) = \overrightarrow{AD}$$

$$(-\tfrac{1}{2}\cdot 4)\overrightarrow{AB} = -2\overrightarrow{AB} = \overrightarrow{AD} .$$

Figure 2d shows an instance of the associative law in which $r = -\frac{1}{2}$, $s = 4$.

VI. The Distributive Laws:

$$r(\overrightarrow{AB} + \overrightarrow{CD}) = r\overrightarrow{AB} + r\overrightarrow{CD} ,$$

$$(r + s)\overrightarrow{AB} = r\overrightarrow{AB} + s\overrightarrow{AB} .$$

$$\overrightarrow{AQ} = 4\overrightarrow{AB} , \overrightarrow{QP} = 4\overrightarrow{CD} , \overrightarrow{AP} = 4\overrightarrow{AD}$$

$$\overrightarrow{AP} = \overrightarrow{AQ} + \overrightarrow{QP}$$

$$4\overrightarrow{AD} = 4\overrightarrow{AB} + 4\overrightarrow{CD}$$

$$4(\overrightarrow{AB} + \overrightarrow{CD}) = 4\overrightarrow{AB} + 4\overrightarrow{CD}$$

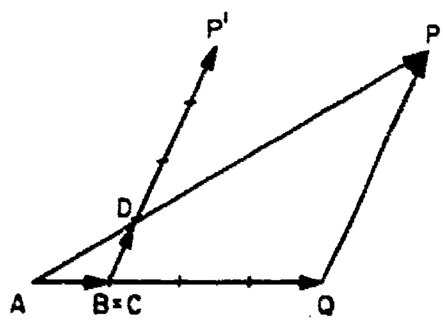


Figure 2e

Figure 2e illustrates the first of the two distributive laws for $r = 4$.

Figure 2f

Figure 2f illustrates the distributive laws for $r = 4$, $s = -2$.

$$\overrightarrow{AD} = \overrightarrow{AC} - \overrightarrow{DC}$$

$$2\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CD}$$

$$(4 + (-2))\overrightarrow{AB} = 4\overrightarrow{AB} + (-2)\overrightarrow{AB}$$

The combined effect of all these laws can be summed up briefly as follows:

> Directed line segments obey the familiar rules of algebra with respect to addition, subtraction, and multiplication by real numbers.

We now show how this algebra of directed line segments can be applied to proving theorems of geometry.

<u>Example 2a</u>. Show that the midpoints of the sides of any quadrilateral are vertices of a parallelogram.

<u>Proof</u>: Let ABCD be the quadrilateral (see Figure 2g)

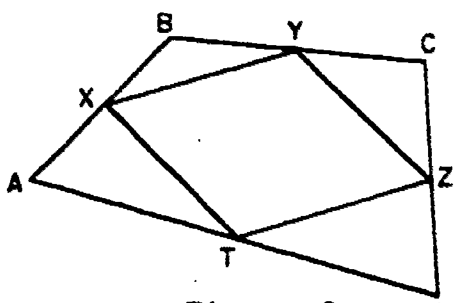


Figure 2g

and let X, Y, Z, T be the midpoints of its sides as indicated. It is sufficient to show that $\overrightarrow{XY} \doteq \overrightarrow{TZ}$ since this implies both that $\overrightarrow{XY} \parallel \overrightarrow{TZ}$ and that $|XY| = |TZ|$.

We have $$\overrightarrow{XY} = \tfrac{1}{2}\overrightarrow{AB} + \tfrac{1}{2}\overrightarrow{BC}$$

and $$\overrightarrow{TZ} = \tfrac{1}{2}\overrightarrow{AD} + \tfrac{1}{2}\overrightarrow{DC}\ .$$

Since $\overrightarrow{DC} = \overrightarrow{DA} + \overrightarrow{AB} + \overrightarrow{BC}$, we also have

$$\overrightarrow{TZ} \doteq \frac{1}{2}\overrightarrow{AD} + \frac{1}{2}(\overrightarrow{DA} + \overrightarrow{AB} + \overrightarrow{BC})$$

$$\doteq \frac{1}{2}\overrightarrow{AD} - \frac{1}{2}\overrightarrow{AD} + \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}$$

$$\doteq \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} .$$

This shows that $\overrightarrow{XY} \doteq \overrightarrow{TX}$.

Example 2b. Prove that the diagonals of a parallelogram bisect each other.

Solution: Let ABCD be the parallelogram (see Figure 2h).

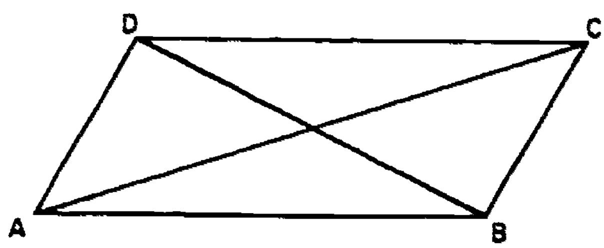


Figure 2h

Then the midpoint of $\overline{AC}$ is the endpoint of $\frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BC})$. The midpoint of $\overline{DB}$ is the endpoint of $\overrightarrow{AB} + \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{AD})$ which equals $\overrightarrow{AB} - \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AD}$ or $\frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AD}$. We show that this is the same as $\frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}$. Since ABCD is a parallelogram, $\overrightarrow{AD} \doteq \overrightarrow{BC}$, so the last sum is certainly equivalent to $\frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}$. We conclude that these directed line segments are the same by noticing that in addition to being equivalent they also have the same initial point.

Example 2c. Prove that the medians of a triangle meet in a point which trisects each of them.

<u>Solution:</u> Let ABC be the triangle (see Figure 21).

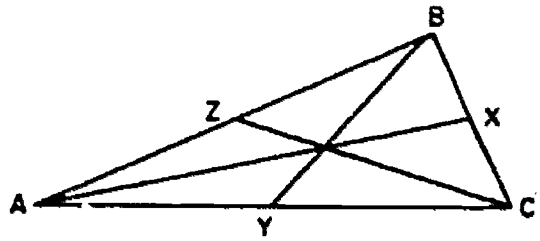


Figure 21

Let X, Y, Z be the midpoints of its sides. Then, the point two-thirds the way from A to X is the endpoint $\frac{2}{3}(\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC})$.
The point two-thirds the way from B to Y is the endpoint of

$$\overrightarrow{AB} + \frac{2}{3}(\overrightarrow{BA} + \frac{1}{2}\overrightarrow{AC}) .$$

The point two-thirds the way from C to Z is the endpoint of

$$\overrightarrow{AC} + \frac{2}{3}(\overrightarrow{CA} + \frac{1}{2}\overrightarrow{AB}) .$$

We show that these three directed line segments are one and the same. We use the fact that $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$.

Then the first is equal to

$$\frac{2}{3}(\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BA} + \frac{1}{2}\overrightarrow{AC})$$

which is equal to $\frac{2}{3}(\overrightarrow{AB} - \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC})$

or $\frac{1}{3}\overrightarrow{AB} + \frac{1}{3}\overrightarrow{AC}$.

The second is equal to $\overrightarrow{AB} - \frac{2}{3}\overrightarrow{AB} + \frac{1}{3}\overrightarrow{AC}$ which also equals

$$\frac{1}{3}\overrightarrow{AB} + \frac{1}{3}\overrightarrow{AC} .$$

The third is equal to $\overrightarrow{AC} - \frac{2}{3}\overrightarrow{AC} + \frac{1}{3}\overrightarrow{AB}$ which also equals

$$\frac{1}{3}\overrightarrow{AC} + \frac{1}{3}\overrightarrow{AB} .$$

<u>Example</u> <u>2d</u>. Prove that the line which joins one vertex of a parallelogram to the midpoint of an opposite side is trisected by a diagonal. Prove also that it trisects this diagonal.

<u>Solution</u>: Let ABCD be the parallelogram (see Figure 2j). Let A be the given vertex and let X be the midpoint of $\overline{CD}$.

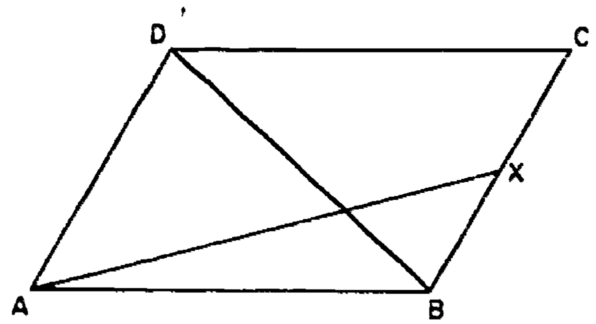


Figure 2j

We are to show that the point two-thirds of the way from A to X is the same as the point two-thirds of the way from D to B.

The first point is the endpoint of

$$\tfrac{2}{3}(\overrightarrow{AB} + \tfrac{1}{2}\overrightarrow{BC})$$

or

$$\tfrac{2}{3}\overrightarrow{AB} + \tfrac{1}{3}\overrightarrow{BC} .$$

The second point is the endpoint of

$$\overrightarrow{AD} + \tfrac{2}{3}(\overrightarrow{DA} + \overrightarrow{AB}) .$$

This latter equals

$$\overrightarrow{AD} - \tfrac{2}{3}\overrightarrow{AD} + \tfrac{2}{3}\overrightarrow{AB}$$

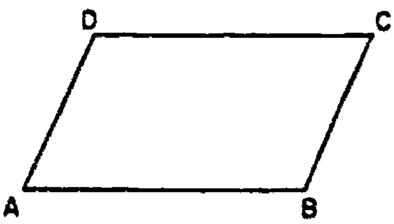
$$\tfrac{1}{3}\overrightarrow{AD} + \tfrac{2}{3}\overrightarrow{AB} .$$

Since $\overrightarrow{AD}$ is equivalent to $\overrightarrow{BC}$ we see that these two directed line segments are equivalent; that they are in fact the same follows from the additional fact that they have the same initial point.

Exercises 2

1. If ABCD is a parallelogram, express $\overrightarrow{DB}$.

   (a) in terms of $\overrightarrow{DC}$ and $\overrightarrow{DA}$.
   (b) in terms of $\overrightarrow{DC}$ and $\overrightarrow{CB}$.
   (c) in terms of $\overrightarrow{AB}$ and $\overrightarrow{BC}$.
   (d) in terms of $\overrightarrow{AB}$ and $\overrightarrow{AD}$.
   (e) in terms of $\overrightarrow{BA}$ and $\overrightarrow{BC}$.



2. If A and B are distinct points, identify the set of all terminal points of the directed line segments of the form $t\ \overrightarrow{AB}$ for which

   (a) $t \geq 0$.
   (b) $0 \leq t \leq 1$.
   (c) $t \geq 1$.
   (d) $-1 \leq t \leq 1$.

3. If A, B, C are non-collinear points, find the set of all terminal points of directed line segments of the form

$$r\ \overrightarrow{AB} + s\ \overrightarrow{AC}$$

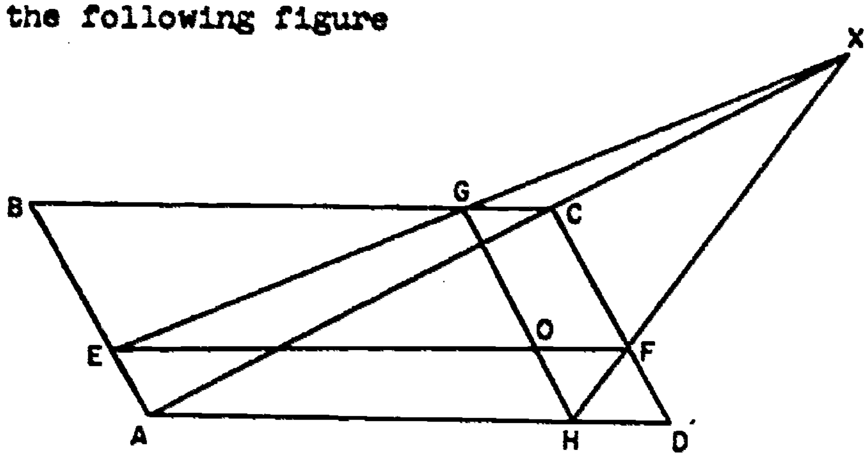
   for which

   (a) $r = 0$, s arbitrary.
   (b) $s = 0$, r arbitrary.
   (c) $0 \leq r \leq 1$, s arbitrary.
   (d) $0 \leq s \leq 1$, r arbitrary.
   (e) $0 \leq r \leq 1$, $0 \leq s \leq 1$.
   (f) $r = 1$, s arbitrary.
   (g) $s = 1$, r arbitrary.
   *(h) $r + s = 1$.
   *(i) $r - s = 1$.
   *(j) $\frac{r}{2} + \frac{s}{3} = 1$.
   *(k) $6r + 7s = 8$.
   *(l) $ar + bs + c = 0$, where $\underline{a}$, $\underline{b}$, $\underline{c}$ are real numbers and where not both $\underline{a}$ and $\underline{b}$ are zero.

4. Show by an example that subtraction of directed line segments

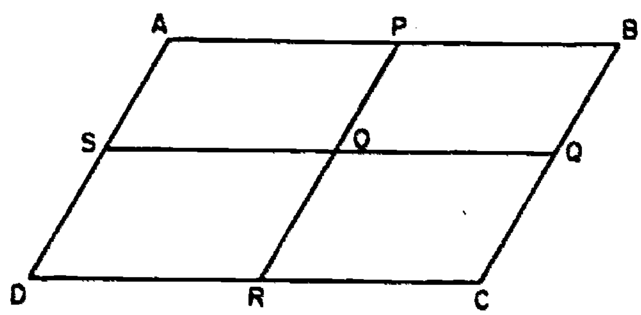
   (a) is not commutative,
   (b) is not associative.

5. In the following figure



ABCD , EOGB , and HDFO are each parallelograms. Prove that their respective diagonals $\overline{AC}$, $\overline{EG}$, $\overline{HF}$, extended if necessary, meet in a single point X .

6. ABCD is a parallelogram and P, Q, R, S are the midpoints of the sides.



For each of the following directed line segments, find an equivalent directed line segment of the form r $\overrightarrow{OQ}$ + s $\overrightarrow{OP}$.

(a) $\overrightarrow{OB}$.
(b) $\overrightarrow{OC}$.
(c) $\overrightarrow{OD}$.
(d) $\overrightarrow{OA}$.
(e) $\overrightarrow{DB}$.
(f) $\overrightarrow{AC}$.
(g) $\overrightarrow{CA}$.
(h) $\overrightarrow{BD}$.

7. Show that the four diagonals of a parallelepiped bisect one another.

3. Vectors and Scalars: Components.

When we define, as in Section 1, operations for the set of directed line segments, this set will be called a set of vectors. The real numbers which we use as multipliers for these vectors will be called scalars.

Neither the nature of the directed segment nor that of the real number has been changed. They are now, however, all seen as parts of a larger entity, a vector space. It is relative to this new system that they are being renamed. From now on we shall call a directed line segment a vector. We shall call a real number a scalar if and when it multiplies a vector.

We are going to discuss equivalence of vectors, addition of vectors and multiplication of vectors by scalars in terms of coordiantes. The following theorem is the basic tool in this discussion.

THEOREM 3a. Let A, B, C, D have respective coordinates (a_1,a_2) , (b_1,b_2) , (c_1,c_2) , (d_1,d_2) . Then

$$\overrightarrow{AB} \doteq \overrightarrow{CD}$$

if and only if

$$b_1 - a_1 = d_1 - c_1 \ , \ b_2 - a_2 = d_2 - c_2 \ .$$

Proof: Figure 3a illustrates Theorem 3a.

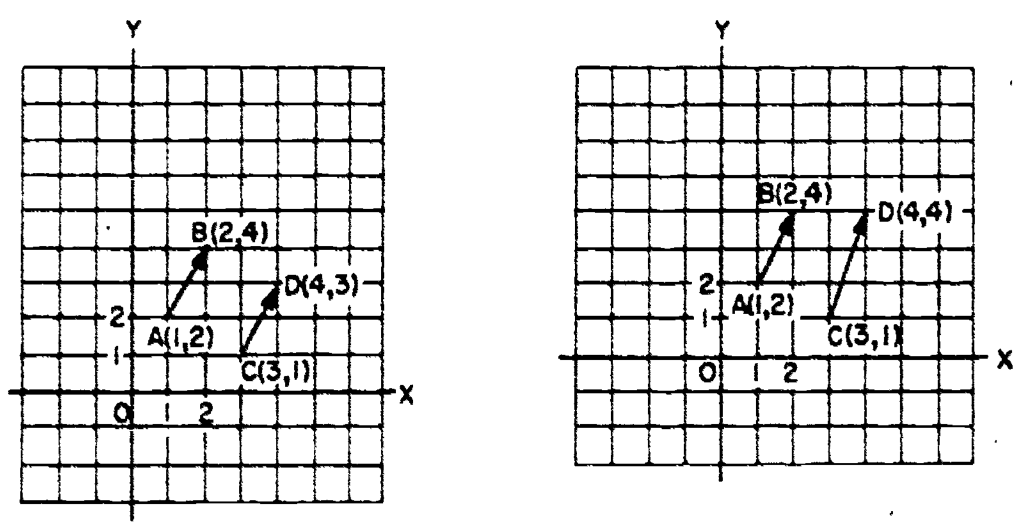


Figure 3a

$\overrightarrow{AB}$ is equivalent to $\overrightarrow{CD}$

$$2 - 1 = 4 - 3$$
$$4 - 2 = 3 - 1$$

$\overrightarrow{AB}$ is not equivalent to $\overrightarrow{CD}$

$$2 - 2 \neq 4 - 1$$

We give only a few indications of the proof of this theorem.

If $b_1 - a_1 = d_1 - c_1$ and if $b_2 - a_2 = d_2 - c_2$ then

$$(b_1 - a_1)^2 + (b_2 - a_2)^2 = (d_1 - c_1)^2 + (d_2 - c_2)^2$$

and

$$\frac{b_2 - a_2}{b_1 - a_1} = \frac{d_2 - c_2}{d_1 - c_1}$$

provided that

$$b_1 - a_1 \neq 0 \text{ and } d_1 - c_1 \neq 0 .$$

We conclude that $|AB| = |CD|$ and that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. This makes plausible the fact that if the given equations hold then $\overrightarrow{AB} \doteq \overrightarrow{CD}$. It doesn't completely prove this (we need $\overrightarrow{AB} \parallel \overrightarrow{CD}$) and it doesn't contribute at all to the proof of the converse.

Corollary. If $\overrightarrow{OP}$ is the vector equivalent to $\overrightarrow{AB}$, where O is the origin, then P has coordinates $(b_1 - a_1 , b_2 - a_2)$.

DEFINITION 3a. If A is the point (a_1,a_2) and B is the point (b_1,b_2), we call the number $b_1 - a_1$ the x-component of $\overrightarrow{AB}$, the number $b_2 - a_2$ the y-component of $\overrightarrow{AB}$.

In most discussions of vectors the initial and terminal points of the vectors are not as important as their x and y-components. We shall therefore often specify a vector by giving its x and y component. We use square brackets [,] to do this; $[p,q]$ means any vector whose x-component is p and whose y-component is q. We shall sometimes denote vectors by single letters, with an arrow above, like $\vec{A}$, when the specific endpoints are not important. We also write $\vec{A} = \vec{B}$ to assert that two vectors are equivalent. The equal sign should properly connect not the vectors themselves but their components. Thus Theorem 3a can be restated as follows:

If $\vec{X}$ is $[x_1,x_2]$ and $\vec{Y}$ is $[y_1,y_2]$, then

$$\vec{X} = \vec{Y}$$

if and only if

$$x_1 = y_1 \quad \text{and} \quad x_2 = y_2 \; .$$

We use the symbol $|X|$ to denote the length of X . We have

$$|[x_1,x_2]| = \sqrt{x_1{}^2 + x_2{}^2} \; .$$

We turn now to the addition and multiplication operations for vectors, show how they can be effected in terms of components, and prove the basic algebraic laws stated for them in Section 2.

THEOREM 3b. If $\vec{X}$ is $[x_1,x_2]$ and if $\vec{Y}$ is $[y_1,y_2]$ then $\vec{X} + \vec{Y}$ is $[(x_1 + x_2) , (y_1 + y_2)]$.

Proof: By definition of addition for vectors (see Figure 3b)

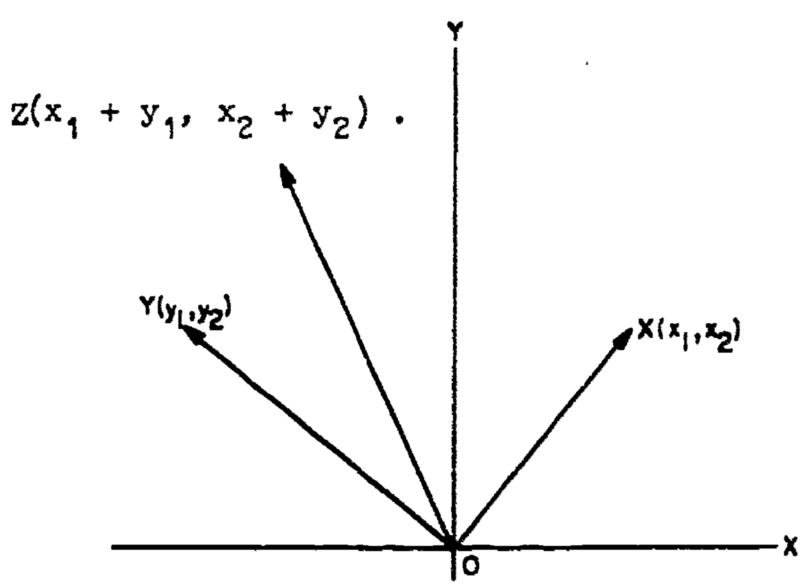


Figure 3b

$\vec{OZ}$ is $\vec{OX} + \vec{OY}$ if and only if $\vec{XZ} \doteq \vec{OY}$. According to Theorem 3a, this will be so if and only if the point Z is $(x_1 + y_1 , x_2 + y_2)$. It follows that the components of $\vec{X} + \vec{Y}$ are $x_1 + y_1$ and $x_2 + y_2$.

<u>Corollary</u>. Addition of vectors is commutative.

$$\vec{X} + \vec{Y} = \vec{Y} + \vec{X} .$$

<u>Corollary</u>. Addition of vectors is associative.

$$(\vec{X} + \vec{Y}) + \vec{Z} = \vec{X} + (\vec{Y} + \vec{Z}) .$$

<u>Corollary</u>. There is a zero vector $[0,0]$.

<u>Corollary</u>. Every vector $\vec{X}$ has an additive inverse $-\vec{X}$. If $\vec{X}$ is $[x_1,x_2]$, then $-\vec{X}$ is $[-x_1,-x_2]$.

<u>THEOREM 3c</u>. If $\vec{X}$ is $[x_1,x_2]$, then $r\vec{X}$ is $[rx_1,rx_2]$.

<u>Proof</u>: Let Y be the point (rx_1,rx_2) , (see Figure 3c).

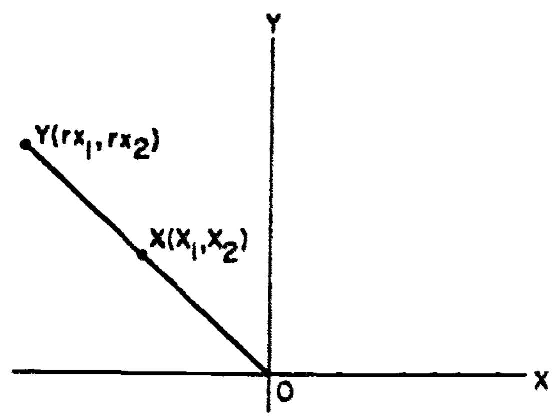


Figure 3c

Then

$$|OY| = \sqrt{(rx_1)^2 + (rx_2)^2} = |r|\sqrt{x_1^{\ 2} + x_2^{\ 2}} = |r|\ |OX| .$$

Also O, X, Y are collinear, since they are on the line whose equation is $x_2x - x_1y = 0$. We must show that the ray $\overrightarrow{OX}$ is parallel to the ray $\overrightarrow{OY}$ to complete our proof. We omit this part of the proof.

<u>Corollary</u>. Multiplication by scalars is associative.

$$r(s\vec{X}) = (rs)\vec{X} .$$

<u>Corollary</u>. Multiplication by scalars obeys the distributive laws.

$$r(\vec{X} + \vec{Y}) = r\vec{X} + r\vec{Y}$$

$$(r + s)\vec{X} = r\vec{X} + s\vec{X} .$$

<u>Corollary</u>. $(-1)\vec{X} = -\vec{X}$.

<u>Corollary</u>. If $\vec{X}$ is $[x_1, x_2]$ and if $\vec{Y}$ is $[y_1, y_2]$ then $r\vec{X} + s\vec{Y}$ is $[rx_1 + sy_1 , rx_2 + sy_2]$.

<u>DEFINITION 3b</u>. Non-zero vectors $\vec{X}$ and $\vec{Y}$ are said to be <u>parallel</u> if and only if the directed line segments $\overrightarrow{OX}$ and $\overrightarrow{OY}$ equivalent to them are collinear.

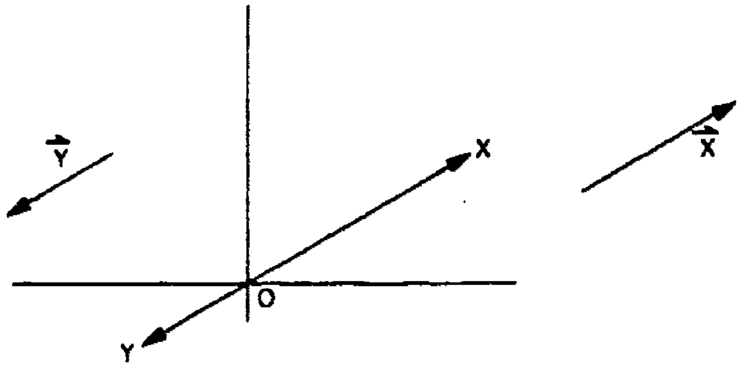


Figure 3c

<u>THEOREM 3d</u>. Non-zero vectors $\vec{X}$ and $\vec{Y}$ are parallel if and only if

$$\vec{Y} = r\vec{X}$$

for some non-zero real number r .

<u>Proof</u>: Let $\vec{X}$ be $[x_1, x_2]$ and $\vec{Y}$ be $[y_1, y_2]$; let X be the point (x_1, x_2) and Y be the point (y_1, y_2) . Then $\overrightarrow{OX} \doteq \vec{X}$, $\overrightarrow{OY} \doteq \vec{Y}$. Then $\vec{X} \parallel \vec{Y}$ if and only if O, X, and Y are collinear. But

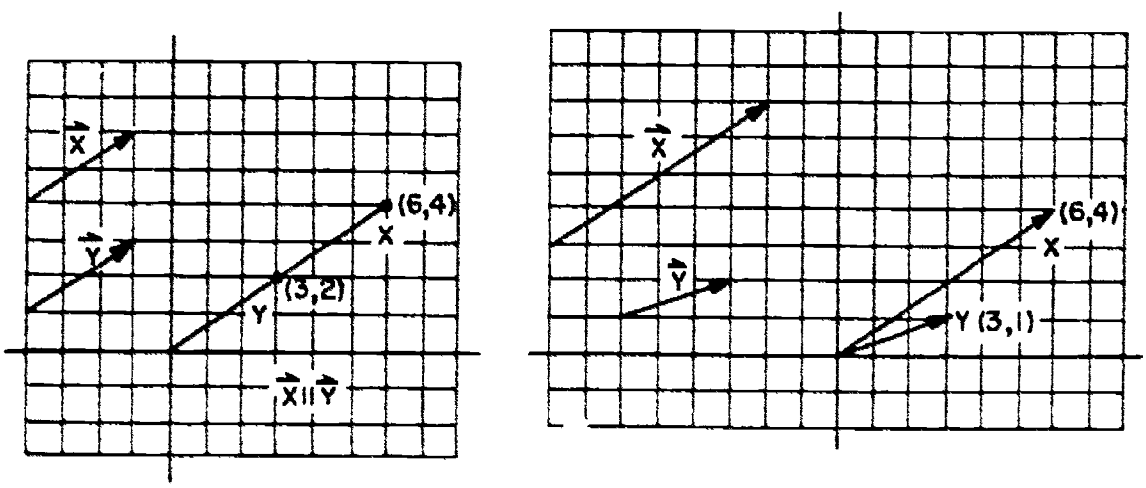
$$\overrightarrow{OX} = r\overrightarrow{OY}$$

if and only if

$$x_1 = ry_1$$

$$x_2 = ry_2$$

and this holds if and only if O, X, Y are collinear.



x not parallel to y

Figure 3d

<u>THEOREM</u> <u>3e</u>. Let $\vec{X}$ and $\vec{Y}$ be any pair of non-zero, non-parallel vectors. Then for each vector $\vec{Z}$ there are numbers r and s such that

$$\vec{Z} = r\vec{X} + s\vec{y} .$$

<u>Proof</u>: Let $\vec{X}$, $\vec{Y}$, $\vec{Z}$ be $[x_1, x_2]$, $[y_1, y_2]$, $[z_1, z_2]$ respectively. We are to show that the equations for r , s

$$z_1 = rx_1 + sy_1$$

$$z_2 = rx_2 + sy_2$$

have a unique solution (r,s) . Since $\vec{X}$ is not parallel to $\vec{Y}$, it follows from Theorem 3d that

$$x_1y_2 - y_1x_2 \neq 0 .$$

<u>Corollary</u>. If $r\vec{X} + s\vec{Y} = \vec{0}$ (where $\vec{0}$ is a zero vector), and $\vec{X}$ and $\vec{Y}$ are non-zero, non-parallel vectors, then

$$r = s = 0 .$$

<u>DEFINITION</u> <u>3c</u>. Any two non-zero, non-parallel vectors in the plane could serve as a <u>basis</u> for all the vectors of the plane.

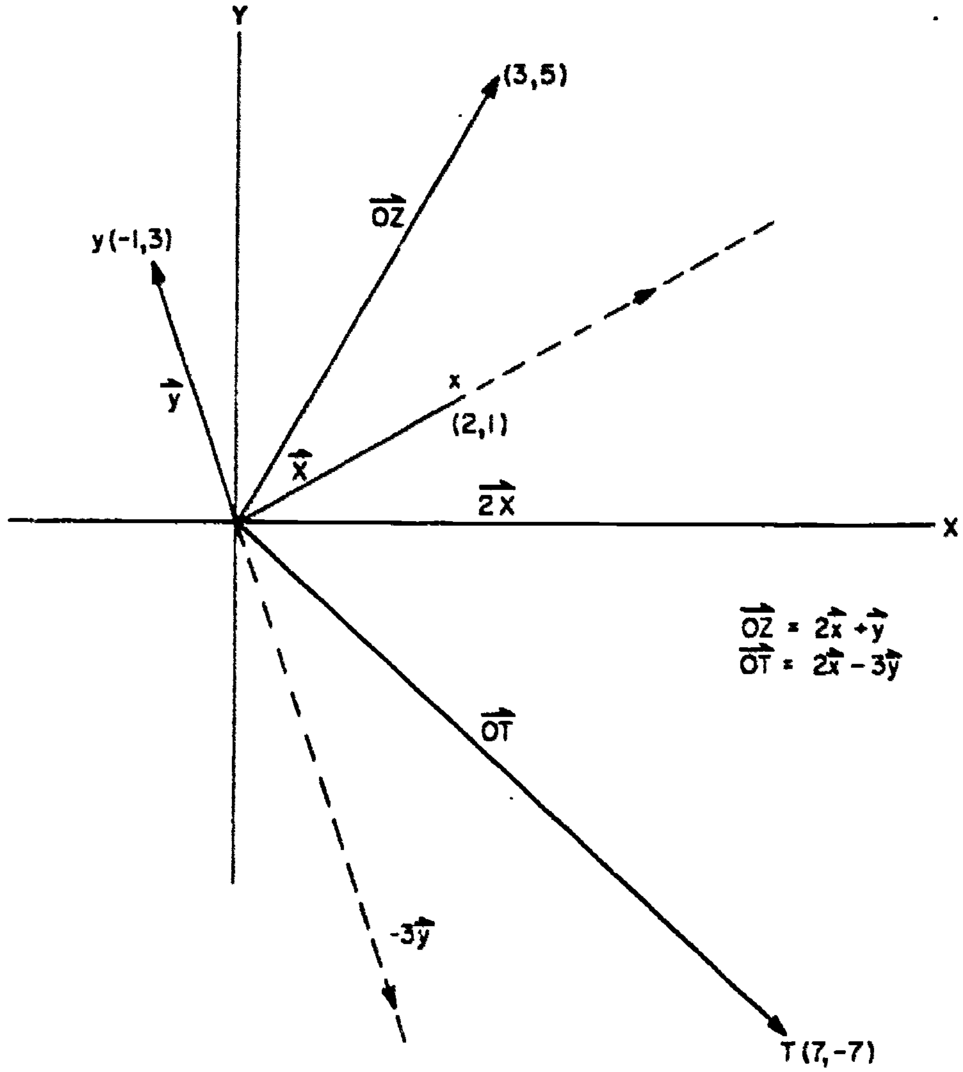


Figure 3e

Figure 3e shows two base vectors $\vec{X}$ and $\vec{Y}$ and vectors $\overrightarrow{OZ}$ and $\overrightarrow{OT}$ expressed in the form $r\vec{X} + s\vec{Y}$.

[1,0] and [0,1] form a base which is frequently used. The vector [1,0] is denoted by $\vec{i}$ and the vector [0,1] is denoted by $\vec{j}$.

<u>THEOREM</u> <u>3f</u>. $\vec{X} = a\vec{i} + b\vec{y}$ if and only if $\vec{X}$ is [a,b] and (a,b) is the point P for which

$$\vec{X} = \overrightarrow{OP} .$$

<u>Proof</u>: If $\vec{X}$ is [a,b] , then, since

$$[a,b] = a[1,0] + b[0,1]$$

it follows that

$$\vec{X} = a\vec{i} + b\vec{j} .$$

If $\vec{X} + a\vec{i} + b\vec{j}$, then

$$\vec{X} = a[1,0] + b[0,1] = [a,b] .$$

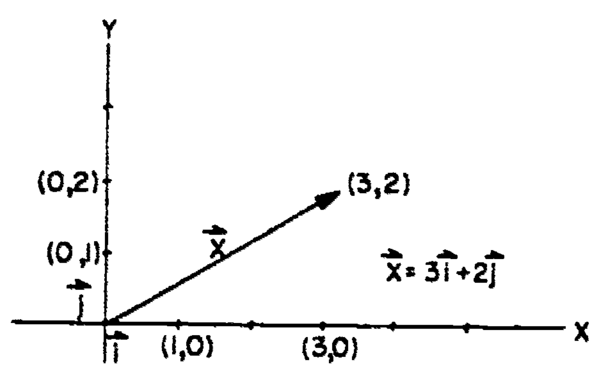


Figure 3f

Figure 3f shows an example of a vector $\vec{X}$ expressed as a sum $3\vec{i} + 2\vec{j}$.

Exercises 3

1. If A, B, and C are respectively $(1,2)$, $(4,3)$, $(6,1)$ find X so that

   (a) $\overrightarrow{AB} \doteq \overrightarrow{CX}$.
   (b) $\overrightarrow{AX} \doteq \overrightarrow{CB}$.
   (c) $\overrightarrow{XA} \doteq \overrightarrow{CB}$.
   (d) $\overrightarrow{XA} \doteq \overrightarrow{BC}$.

2. Same as Problem 1, if A, B, C are respectively $(-1,2)$, $(4,-3)$, $(-6,-1)$.

3. Find the components of

   (a) $[3,2] + [4,1]$.
   (b) $[3,-2] + [-4,1]$.
   (c) $4[5,6]$.
   (d) $-4[5,6]$.
   (e) $-1[5,6]$.
   (f) $-[5,6]$.
   (g) $3[4,1] + 2[-1,3]$.
   (h) $3[4,1] - 2[-1,3]$.

4. Determine x and y so that

   (a) $x[3,-1] + y[3,1] = [5,6]$.
   (b) $x[3,2] + y[2,3] = [1,2]$.
   (c) $x[3,2] + y[-2,3] = [5,6]$.
   (d) $x[3,2] + y[6,4] = [-3,-2]$ (infinitely many solutions).

5. Determine a and b so that

   (a) $[3,1] = a\vec{i} + b\vec{j}$.
   (b) $[1,-3] = a\vec{i} + b\vec{j}$.
   (c) $\vec{i} = a[-3,1] + b[1,-3]$.
   (d) $\vec{j} = a[-3,1] + b[1,-3]$.

6. Determine a and b so that
   $3\vec{i} - 2\vec{j} = a(3\vec{i} + 4\vec{j}) + b(4\vec{i} + 3\vec{j})$.

---

4. Inner Product.

Our algebra of vectors does not yet include multiplication of one vector by another. We now define such a product.

We first say what we mean by the angle between two vectors $\vec{X}$ and $\vec{Y}$ which do not necessarily have a common initial point.

DEFINITION 4a. Let $\vec{X}$ and $\vec{Y}$ be any non-zero vectors and let $\overrightarrow{OX}$ and $\overrightarrow{OY}$ be vectors whose initial point is the origin O and which are equivalent respectively to $\vec{X}$ and $\vec{Y}$. Then by the angle between $\vec{X}$ and $\vec{Y}$ we mean the angle between $\overrightarrow{OX}$ and $\overrightarrow{OY}$.

DEFINITION 4b. Let $\vec{X}$ and $\vec{Y}$ be any vectors. Then the inner product of $\vec{X}$ and $\vec{Y}$ is the real number

$$|\vec{X}|\,|\vec{Y}| \cos \theta$$

where $|\vec{X}|$ is the length $\vec{X}$, $|\vec{Y}|$ is the length of $\vec{Y}$ and θ is the angle between $\vec{X}$ and $\vec{Y}$. (If $\vec{X}$ or $\vec{Y}$ is a zero vector then θ is not defined. We interpret the definitions to mean that the inner product is zero, in this case.)

The inner product has important properties. Before we investigate these properties of the inner product we relate the inner product to a familiar mathematical relation--the law of cosines.

If our given vectors $\vec{X}$ and $\vec{Y}$ are not parallel they determine a triangle OXY, where O is the origin and where X and Y are endpoints of the vectors $\overrightarrow{OX}$ and $\overrightarrow{OY}$ respectively equivalent to $\vec{X}$ and $\vec{Y}$. We can find at least one earlier appearance of the inner product by applying the law of cosines to the triangle. It asserts (Figure 4a)

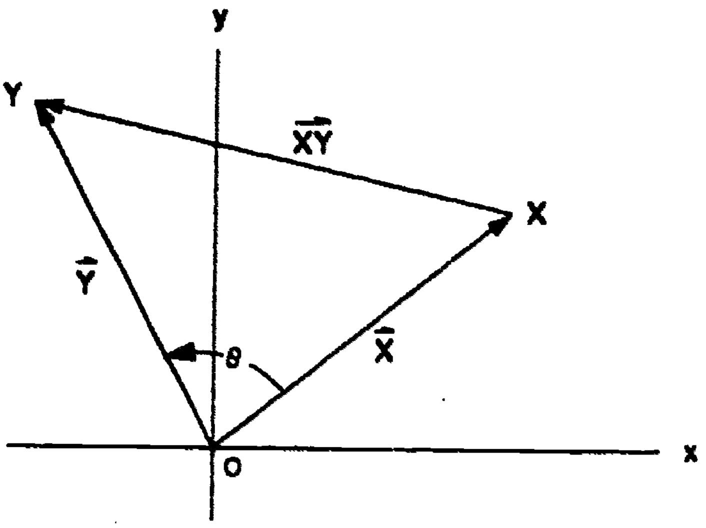


Figure 4a

$$|\overrightarrow{XY}|^2 = |\overrightarrow{OX}|^2 + |\overrightarrow{OY}|^2 - 2|\overrightarrow{OX}| \cdot |\overrightarrow{OY}| \cos \theta$$

whence

$$|\overrightarrow{OX}| \cdot |\overrightarrow{OY}| \cos \theta = \frac{|\overrightarrow{OX}|^2 + |\overrightarrow{OY}|^2 - |\overrightarrow{XY}|^2}{2} .$$

Thus the expression we have called the "inner product" is suggested by the law of cosines.

We sometimes denote this product by the symbols $\vec{X} \cdot \vec{Y}$ (read "$\vec{X}$ dot $\vec{Y}$") and sometimes call it the "dot product."

Usually, in algebra, a multiplication operation for a set of objects assigns a member of this set to each pair of its members. The inner product is not an operation of this type. It does not assign a vector to a pair of vectors but rather it assigns a real number to each pair of vectors.

<u>Example</u> <u>4a</u>. Evaluate $\vec{X} \cdot \vec{Y}$ if $|\vec{X}| = 2$, $|\vec{Y}| = 3$, and (a) $\theta = 0$, (b) $\theta = 45^\circ$, (c) $\theta = 90^\circ$, (d) $\theta = 180^\circ$.

Solution:

(a) $\vec{X} \cdot \vec{Y} = 2 \cdot 3 \cos 0^\circ = 2 \cdot 3 \cdot 1 = 6$.

(b) $\vec{X} \cdot \vec{Y} = 2 \cdot 3 \cos 45^\circ = 2 \cdot 3 \cdot \frac{\sqrt{2}}{2} = 3\sqrt{2}$.

(c) $\vec{X} \cdot \vec{Y} = 2 \cdot 3 \cos 90^\circ = 2 \cdot 3 \cdot 0 = 0$.

(d) $\vec{X} \cdot \vec{Y} = 2 \cdot 3 \cos 180^\circ = 2 \cdot 3 \cdot (-1) = -6$.

The inner product has many applications. One of these is a test for perpendicularity.

THEOREM 4a. If $\vec{X}$ and $\vec{Y}$ are non-zero vectors, then they are perpendicular if and only if

$$\vec{X} \cdot \vec{Y} = 0 .$$

Proof: According to the definition of inner product

$$\vec{X} \cdot \vec{Y} = |\vec{X}| \cdot |\vec{Y}| \cos \theta \quad .$$

This product of real numbers is zero if and only if one of its factors is zero. Since $\vec{X}$ and $\vec{Y}$ are non-zero vectors, the numbers $|\vec{X}|$ and $|\vec{Y}|$ are not zero. Therefore the product is zero if and only if $\cos \theta = 0$, which is the case if and only if $\vec{X}$ and $\vec{Y}$ are perpendicular.

The following theorem supplies a useful formula for the inner product of vectors.

THEOREM 4b. If $\vec{X} = [x_1, x_2]$, $\vec{Y} = [y_1, y_2]$ then

$$\vec{X} \cdot \vec{Y} = x_1 y_1 + x_2 y_2 .$$

Proof: According to the law of cosines (see Figure 4b)

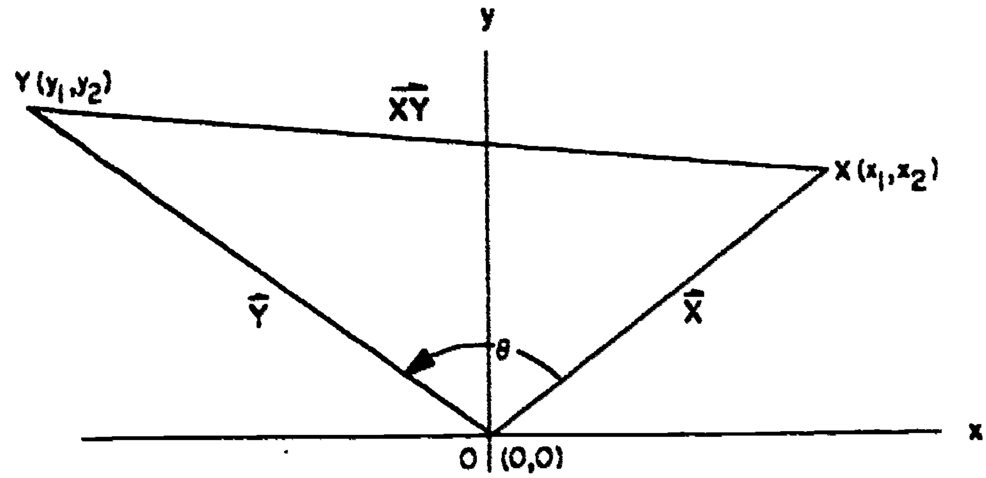


Figure 4b

$$|\vec{OX}| \cdot |\vec{OY}| \cos\theta = \frac{|\vec{OX}|^2 + |\vec{OY}|^2 - |\vec{XY}|^2}{2}$$

$$= \frac{x_1{}^2 + x_2{}^2 + y_1{}^2 + y_2{}^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2}{2}$$

$$= x_1y_1 + x_2y_2 \ .$$

Since, by definition, the left member of this equation is $\vec{X} \cdot \vec{Y}$, our theorem is proved.

Example 4b. If $\vec{X}$ is [3,4] and $\vec{Y}$ is [5,2] , find $\vec{X} \cdot \vec{Y}$.

Solution: $\vec{X} \cdot \vec{Y} = 3 \cdot 5 + 4 \cdot 2 = 23$.

Example 4c. If $\vec{X}$ is [3,7] and $\vec{Y}$ is [-7,3] , show that $\vec{X}$ and $\vec{Y}$ are perpendicular.

Solution: $\vec{X} \cdot \vec{Y} = 3(-7) + 7 \cdot 3 = 0$.

The conclusion follows from Theorem 4a, and the fact that $\vec{X}$ and $\vec{Y}$ are non-zero.

A useful fact about inner products is that they have some of the algebraic properties of products of numbers. The following theorem gives one such common property.

<u>THEOREM 4c</u>. If $\vec{X}$, $\vec{Y}$, $\vec{Z}$ are any vectors, then

$$\vec{X} \cdot (\vec{Y} + \vec{Z}) = \vec{X} \cdot \vec{Y} + \vec{X} \cdot \vec{Z}$$

and

$$(t\vec{X}) \cdot \vec{Y} = t(\vec{X} \cdot \vec{Y}) .$$

<u>Proof</u>: Let $\vec{X} = [x_1, x_2]$, $\vec{Y} = [y_1, y_2]$, $\vec{Z} = [z_1, z_2]$.

Then

$$\begin{aligned} \vec{X} \cdot (\vec{Y} + \vec{Z}) &= [x_1, x_2] \cdot [y_1 + z_1 , y_2 + z_2] \\ &= x_1(y_1 + z_1) + x_2(y_2 + z_2) \\ &= x_1y_1 + x_2y_2 + x_1z_1 + x_2z_2 \\ &= \vec{X} \cdot \vec{Y} + \vec{X} \cdot \vec{Z} \end{aligned}$$

$$\begin{aligned} (t\vec{X}) \cdot \vec{Y} &= [tx_1, tx_2] \cdot [y_1, y_2] \\ &= tx_1y_1 + tx_2y_2 \\ &= t(\vec{X} \cdot \vec{Y}) . \end{aligned}$$

<u>Corollary</u>. $\vec{X} \cdot (a\vec{Y} + b\vec{Z}) = a(\vec{X} \cdot \vec{Y}) + b(\vec{X} \cdot \vec{Z})$.

In certain applications of vectors to physics the notion of a component of a vector in the direction of another vector is important. We now define this concept.

<u>DEFINITION 4c</u>. Let $\vec{X}$ be any non-zero vector and let $\vec{Y}$ be any vector. Then the component of $\vec{Y}$ in the direction of $\vec{X}$ is the number given by each of the following equal expressions:

$$\frac{\vec{X} \cdot \vec{Y}}{|\vec{X}|} = \frac{|\vec{X}| \cdot |\vec{Y}| \cos \theta}{|\vec{X}|} = |\vec{Y}| \cos \theta$$

NOTE: The component of $\vec{Y}$ in the direction of $\vec{X}$ can be described geometrically (see Figure 4c).

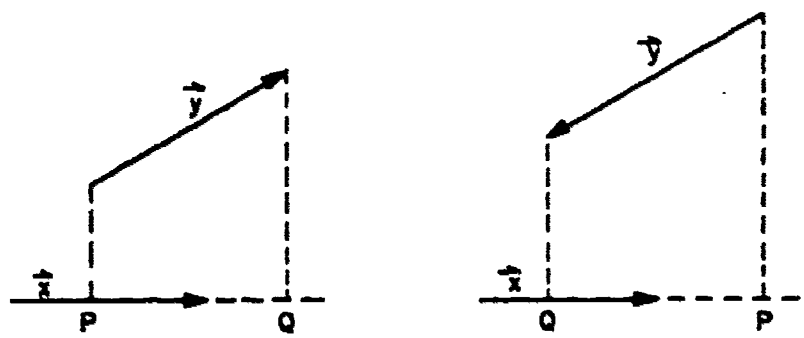


Figure 4c

In both parts of the figure, P is the foot of the perpendicular from the initial point of $\vec{Y}$ to the line of $\vec{X}$, and Q is the foot of the perpendicular from the terminal point of $\vec{Y}$ to this line. In the first part the component of $\vec{Y}$ in the direction of $\vec{X}$ turns out to be the distance from P to Q . In the second part this component turns out to be the negative of the distance from P to Q .

The inner product is used frequently in applications of vectors to physics. For the moment we consider inner products from a purely mathematical standpoint.

Example 4d. Let $\vec{X}$ be any vector parallel to the positive x-axis, let $\vec{Y}$ be any vector parallel to the positive y-axis and let $\vec{Z}$ be the vector $[p,q]$. Show that p and q are the components of $\vec{Z}$ in the direction of $\vec{X}$ and $\vec{Y}$ respectively.

Solution:

$$\cos\theta = \frac{p}{\sqrt{p^2+q^2}}$$

so $$p = (\cos\theta) \cdot \sqrt{p^2+q^2} .$$

Since $|\vec{Z}| = \sqrt{p^2+q^2}$, we conclude that

$$p = |\vec{Z}| \cos\theta .$$

The angle between $\vec{Z}$ and the y-axis is $\frac{\pi}{2} - \theta$.
Consequently the component of $\vec{Z}$ in the direction of $\vec{Y}$ is

$$\cos(\tfrac{\pi}{2} - \theta)\sqrt{p^2 + q^2} .$$

Since $\cos(\frac{\pi}{2} - \theta) = \sin\theta$ and since $\sin\theta = \frac{q}{\sqrt{p^2 + q^2}}$,
we conclude that this component is, in fact,

$$\frac{q}{\sqrt{p^2 + q^2}}\sqrt{p^2 + q^2} \text{ , or } q .$$

<u>Vectors in Three Dimensions</u>.

Much of our discussion of vectors in the plane can be carried over to three dimensions with only minor modifications.

The portions about directed line segments require no modification. When we come to coordinates and components, the conclusions are as follows:

1. The components of a vector in three dimensional space are an ordered triple $[a,b,c]$ of real numbers.
2. Two vectors $[a,b,c]$ and $[p,q,r]$ are equal if and only if $a = p$, $b = q$ and $c = r$.
3. The addition of vectors $[a,b,c]$ and $[p,q,r]$ is given by the rule
$$[a,b,c] + [p,q,r] = [a + p , b + q , c + r] .$$
4. Scalar multiplication of vectors is given by the rule
$$r[a,b,c] = [ra,rb,rc] .$$
5. The unit base vectors associated with the coordinate axes are
$$\vec{i} = [1,0,0]$$
$$\vec{j} = [0,1,0]$$
$$\vec{k} = [0,0,1] .$$

Figure 4d shows these base vectors.

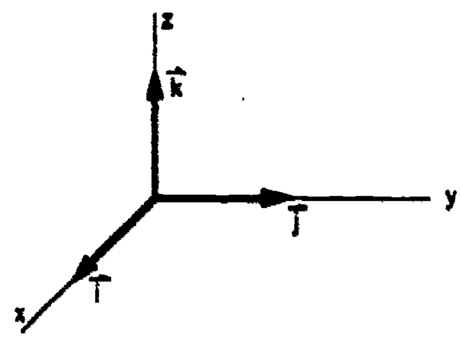


Figure 4d

The vector $\vec{V} = 4\vec{i} + 8\vec{j} + 8\vec{k}$ is illustrated in Figure 4e.

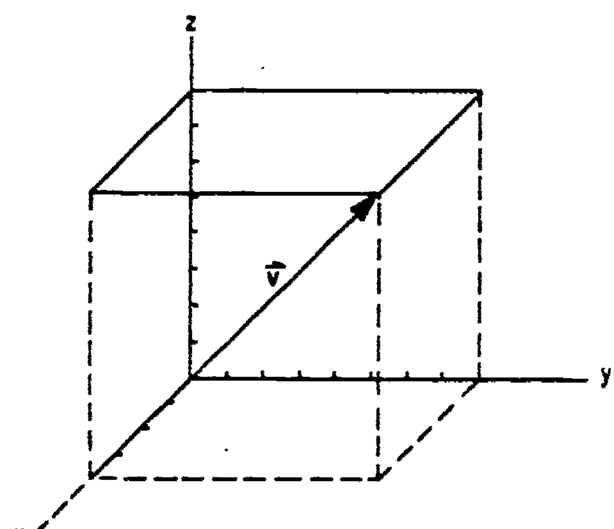


Figure 4e

6. The inner product of $\vec{V}$ and $\vec{W}$ is still given by

$$\vec{V} \cdot \vec{W} = |\vec{V}||\vec{W}| \cos \theta \ .$$

In component form, if $\vec{V}$ is $[v_1, v_2, v_3]$ and $\vec{W}$ is $[w_1, w_2, w_3]$, then

$$\vec{V} \cdot \vec{W} = v_1 w_1 + v_2 w_2 + v_3 w_3 \ ;$$

also $$|\vec{V}| = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2} \ .$$

Exercises 4

1. Find $\vec{X} \cdot \vec{Y}$ if

(a) $\vec{X} = \vec{i}$, $\vec{Y} = \vec{j}$.
(b) $\vec{X} = \vec{i}$, $\vec{Y} = \vec{i}$.
(c) $\vec{X} = \vec{j}$, $\vec{Y} = \vec{i}$.
(d) $\vec{X} = \vec{j}$, $\vec{Y} = \vec{j}$.
(e) $\vec{X} = \vec{i} + \vec{j}$, $\vec{Y} = \vec{i} - \vec{j}$.
(f) $\vec{X} = 2\vec{i} + 3\vec{j}$, $\vec{Y} = 4\vec{i} - 5\vec{j}$.
(g) $\vec{X} = -2\vec{i} - 3\vec{j}$, $\vec{Y} = -4\vec{i} + 5\vec{j}$.
(h) $\vec{X} = a\vec{i} + b\vec{j}$, $\vec{Y} = c\vec{i} + d\vec{j}$.
(i) $\vec{X} = a\vec{i} + b\vec{j}$, $\vec{Y} = 4\vec{X}$.
(j) $\vec{X} = a\vec{i} + b\vec{j}$, $\vec{Y} = s\vec{X}$.

2. Find the angle between $\vec{X}$ and $\vec{Y}$ if $|\vec{X}| = 2$, $|\vec{Y}| = 3$, and $\vec{X} \cdot \vec{Y}$ is

(a) 0 .
(b) 1 .
(c) -2 .
(d) 3 .
(e) -4 .
(f) 5 .
(g) 6 .
(h) -6 .

3. If $\vec{X} = 3\vec{i} + 4\vec{j}$, determine a so that $\vec{Y}$ is perpendicular to $\vec{X}$, if $\vec{Y}$ is

(a) $a\vec{i} + 4\vec{j}$.
(b) $a\vec{i} - 4\vec{j}$.
(c) $4\vec{i} + a\vec{j}$.
(d) $a\vec{i} - 3\vec{j}$.

4. Find the angle between $\vec{X}$ and $\vec{Y}$ in each part of Exercise 1 above.

5. If $a^2 + b^2 \neq 0$ prove that $a\vec{i} + b\vec{j}$ is perpendicular to $c\vec{i} + d\vec{j}$ if and only if $a\vec{i} + b\vec{j}$ is parallel to $-d\vec{i} + c\vec{j}$.

6. Find the component of $\vec{Y}$ in the direction of $\vec{X}$ if

(a) $\vec{X} = \vec{i}$, $\vec{Y} = 3\vec{i} + 4\vec{j}$.
(b) $\vec{X} = \vec{j}$, $\vec{Y} = 3\vec{i} + 4\vec{j}$.
(c) $\vec{X} = 3\vec{i} + 4\vec{j}$, $\vec{Y} = \vec{i}$.
(d) $\vec{X} = 3\vec{i} + 4\vec{j}$, $\vec{Y} = \vec{j}$.
(e) $\vec{X} = 3\vec{i} + 4\vec{j}$, $\vec{Y} = 3\vec{i} + 4\vec{j}$.
(f) $\vec{X} = 3\vec{i} + 4\vec{j}$, $\vec{Y} = 5\vec{i} + 2\vec{j}$.
(g) $\vec{X} = 3\vec{i} + 4\vec{j}$, $\vec{Y} = a\vec{i} + b\vec{j}$.
(h) $\vec{X} = p\vec{i} + q\vec{j}$, $\vec{Y} = a\vec{i} + b\vec{j}$.

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5. Applications of Vectors in Physics.

The notion of "force" is one of the important concepts of physics. This is the abstraction which physicists have invented to describe "pushes" and "pulls" and to account for the effects that pushes and pulls produce.

The student who knows something about vectors can readily learn about forces. The connecting links between the concepts of "force" and "vector" can be stated as follows:

1. Every force can be represented as a vector. The direction of the force is the same as the direction of its representative vector. The magnitude of the force determines the length of its representing vector, once a "scale" has been selected.

Example 5a. A red-headed man is standing on top of a hill, staring into space. He weighs 200 pounds. Represent as a vector each of the following:

(a) the downward pull of the earth's gravity on him,
(b) the upward push of the hill on him.

Solution: (a)

Scale: 1 inch represents 200 pounds

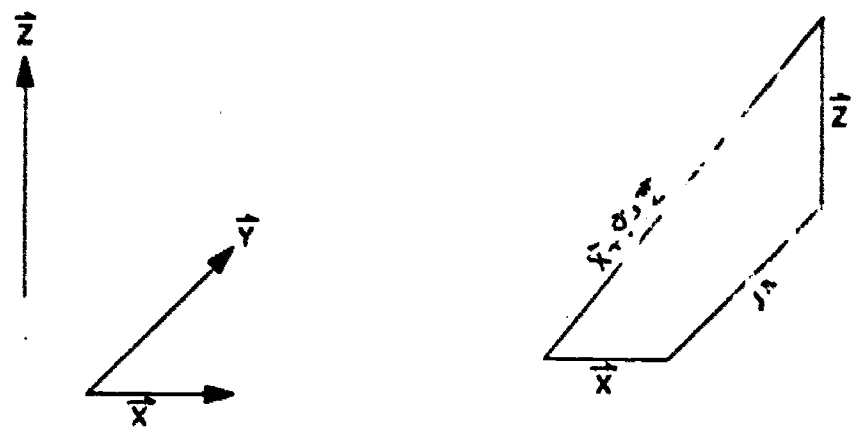
(b)

Scale: 1 inch represents 200 pounds

2. Any collection of forces which act on a single body is equivalent to a single force, called their resultant. If all the forces are represented as vectors on the same scale, then the vector which represents the resultant of the forces is the sum of these vectors.

Example 5b. Represent each of the following forces as a vector, and find the vector which represents their resultant: A force F_1 of 300 pounds directed to the right, a force F_2 of 400 pounds directed at an angle of 45° with the x-axis and a force of 500 pounds directed upward.

<u>Solution:</u> (graphical) Using the scale 1 inch represents 400 pounds, $\vec{X}$ represents F_1, $\vec{Y}$ represents F_2, $\vec{Z}$ represents F_3.



$\vec{X} + \vec{Y} + \vec{Z}$ represents the resultant of F_1, F_2, F_3. Its length is a little less than 5/2 inches; its direction is about 54°.

3. If F and G are two forces which have the same direction, then they have a numerical ratio and there is a number r such that r times force F is equivalent to force G. Moreover if $\vec{F}$ is the vector which represents force F, then $r\vec{F}$ is the vector which represents force G, where r is the ratio of force G to force F.

<u>Example 5c</u>. Emily and Elsie are identical twins. They are sitting on a fence. If $\vec{F}$ represents the total force Emily and Elsie exert on the fence and if $\vec{G}$ represents the force the fence exerts on Emily alone, express

(a) $\vec{F}$ in terms of $\vec{G}$.
(b) $\vec{G}$ in terms of $\vec{F}$.

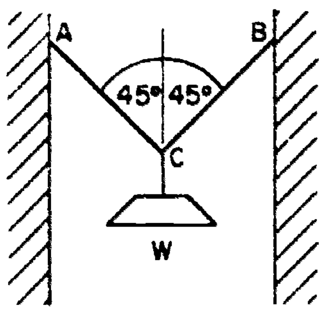
<u>Solution:</u>

(a) $\vec{F} = -2\vec{G}$.
(b) $\vec{G} = -\frac{1}{2}\vec{F}$.

A body at rest is said to be in equilibrium. It is a fact of physics that if a body is at rest the resultant of all the forces acting on the body has magnitude zero. (Note: The converse of this is not true, since the resultant of all the forces which act on a moving body can also be zer . According to the laws of physics, if the sum of all the forc s which act on a body is zero, then the body must be either at rest or it must move in a straight lin with constant speed.)

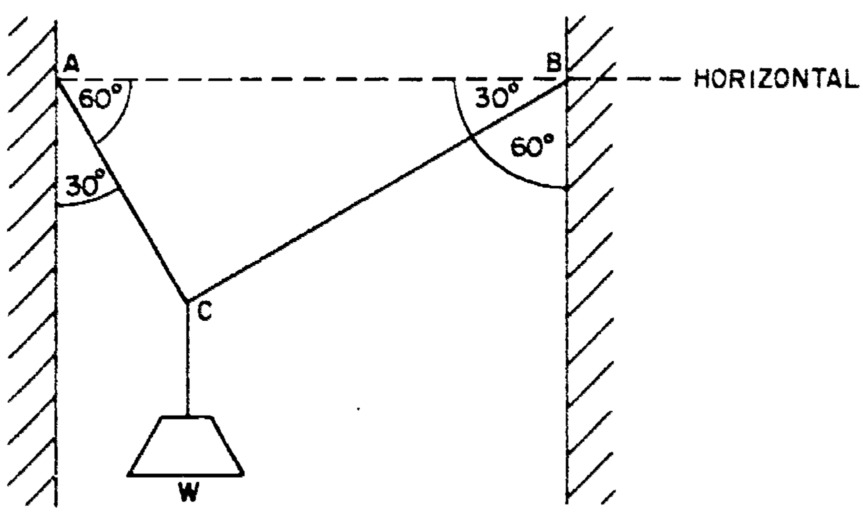
Exercises 5a

1. A weight is suspended by ropes as shown in the figure.



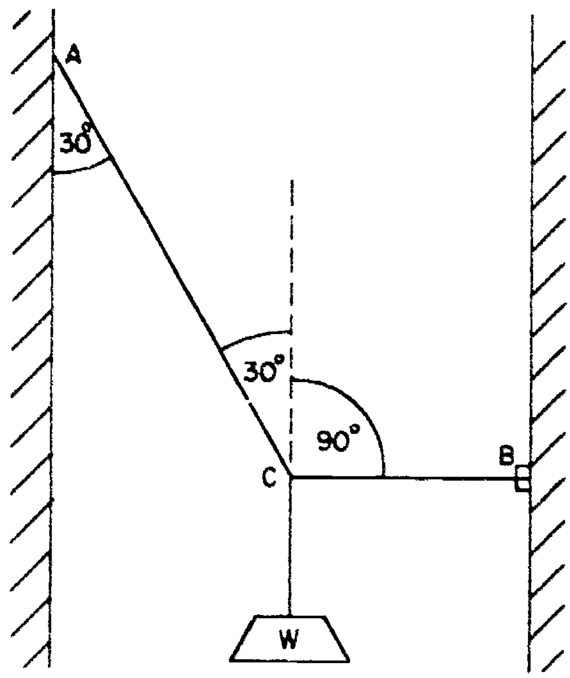
If the weight weighs 10 pounds, what is the force exerted on the junction C by the rope CB ?

2. A weight of 1,000 pounds is suspended from wires as shown in the figure.

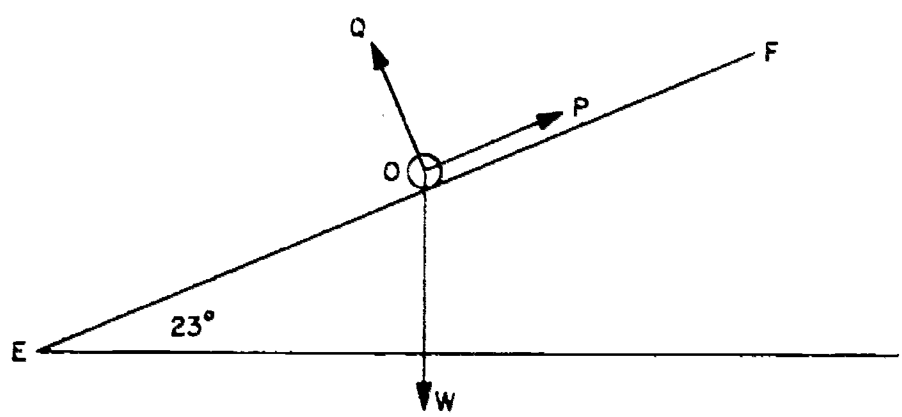


The distance AB is 20 feet. AC is 10 feet, and CB is $10\sqrt{3}$ feet. What force does the wire AC exert on the junction C ? What force does the wire BC exert on C ? If all three wires are about equally strong, which wire is most likely to break? Which wire is least likely to break?

3. A 5,000 pound weight is suspended as shown in the figure. Find the tension in each of the ropes CA , CB , and CW .

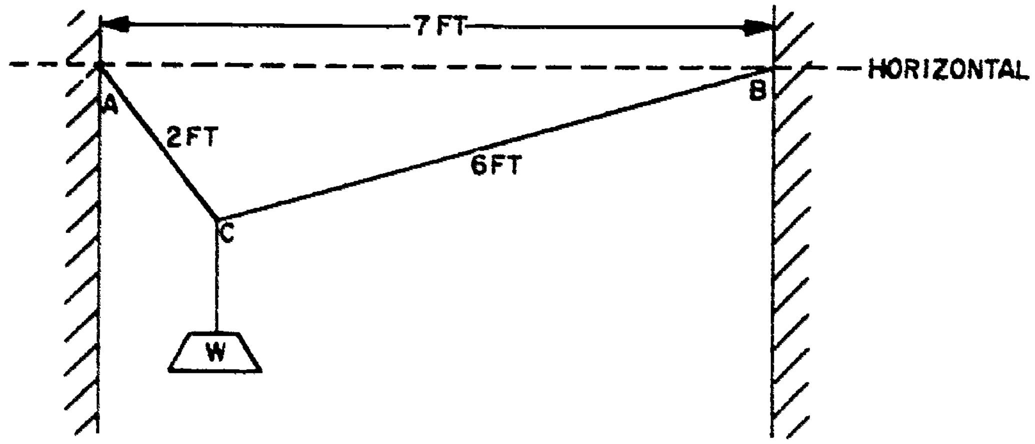


4. A barrel is held in place on an inclined plane EF by a force $\overrightarrow{OP}$ operating parallel to the plane and another operating perpendicular to it. (See diagram.)

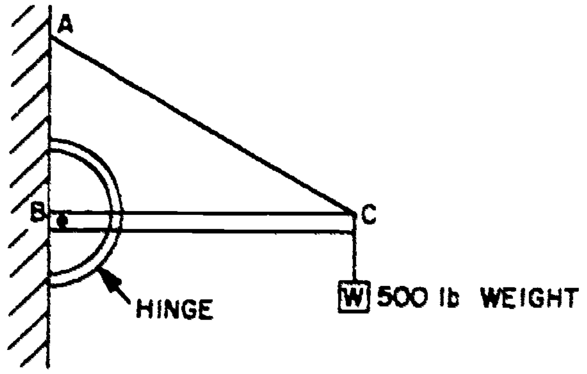


If the weight of the barrel is 300 pounds, ($|\overrightarrow{OW}| = 300$) and the plane makes an angle of 23° with the horizontal, find $|\overrightarrow{OP}|$ and $|\overrightarrow{OQ}|$. (Hint: Introduce a coordinate system with origin at O and OW as negative y-axis.)

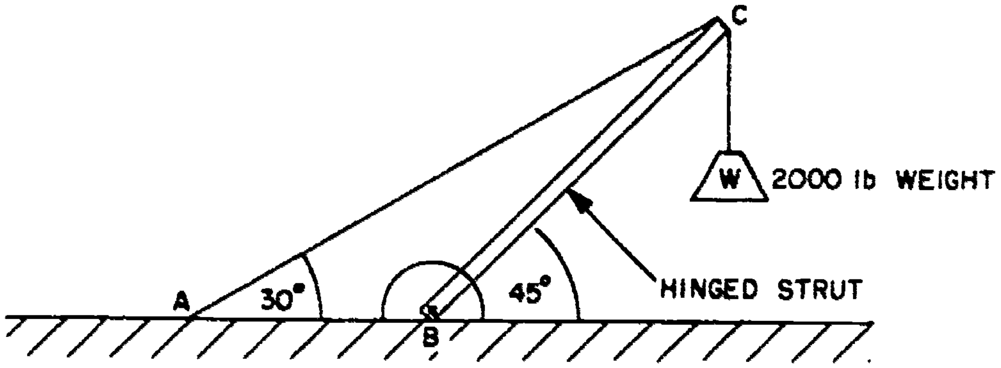
5. A weight is suspended by ropes as shown in the figure. If the weight weighs 20 pounds, what is the force exerted on the junction C by the rope CB ? By the rope AC ? If AC and CB are equally strong, which one is more likely to break?

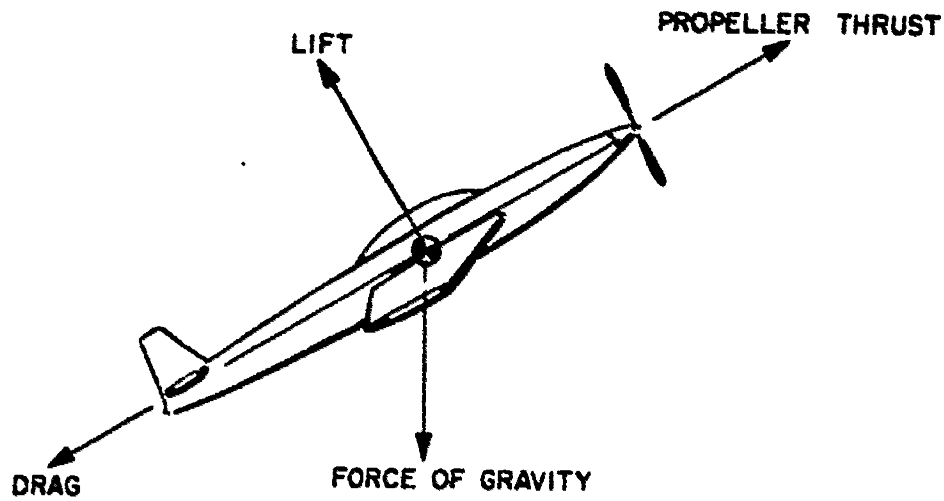


6. A 500 pound weight is suspended as shown in the figure. Find each of the forces exerted on point C .



7. A 2,000 pound weight is lifted at constant speed, as shown in the diagram. Find each of the forces exerted on point C .





The motion of airplanes provides another application of vectors. Some technical terms involved are listed and illustrated in the figure.

<u>Lift</u>: $F_{\vec{L}}$--a force perpendicular to the direction of motion. This is the "lifting force" of the wing.

<u>Gravity</u>: $F_{\vec{g}}$--a force directed downward.

<u>Propeller thrust</u>: $F_{\overrightarrow{pt}}$--a forward force in the direction of motion.

<u>Drag</u>: $F_{\vec{d}}$--a backward force parallel to the motion. This force is due to wind resistance.

<u>Effective propeller thrust</u>: $F_{\overrightarrow{ept}}$--the propeller thrust minus the drag.

The physical principle we shall use states that a body in motion will continue to move in a straight line with constant speed if and only if the resultant of all the forces acting on the body is zero.

8. An airplane weighing 6,000 pounds climbs steadily upwards at an angle of 30°. Find the effective propeller thrust and the lift.

9. An airplane weighing 10,000 pounds climbs at an angle of 15° with constant speed. Find the effective propeller thrust and the lift.

10. A motorless glider descends at an angle of 10° with constant speed. If the glider and occupant together weigh 500 pounds, find the drag and the lift.

The term "work" as the physicists use it also provides an example of a concept which can be discussed in terms of vectors. Consider for instance a tractor pulling a box-car along a track.

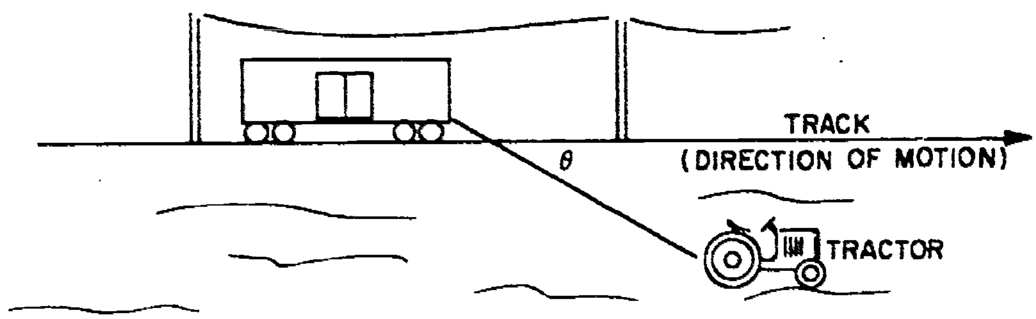


Figure 5a

The effect of the tractor's force depends on the angle θ . It also involves the force itself and the displacement produced. The term "work," as used in physics, is given in this case by $\vec{F} \cdot \vec{S}$, where $\vec{F}$ is the force-vector of the tractor and where $\vec{S}$ is the displacement of the box-car.

More generally, if a force F acts on a body and produces a displacement S while it acts, then the work done by the force is defined to be $\vec{F} \cdot \vec{S}$, where $\vec{F}$ is the vector which represents the force and where $\vec{S}$ is the vector which represents the displacement.

Example 5e. If the tractor of Figure 5a exerts a force of 1,000 pounds at an angle of 30° to the track, how much work does the tractor do in moving a string of cars 2,000 feet?

Solution: Evaluate the expression $|\vec{F}|\ |\vec{S}| \cos \theta$ where $|\vec{F}| = 1{,}000$ pounds, $|\vec{S}| = 2{,}000$ feet, $\cos \theta = .866$. The value of this product is 1,732,000 foot pounds.

Exercises 5b

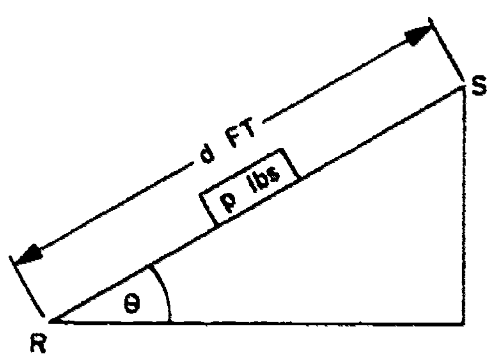
1. A sled is pulled a distance of d feet by a force of p pounds which makes an angle of θ with the horizontal. Find the work done if

   (a) $d = 10$ feet, $p = 10$ pounds, $\theta = 10^\circ$.
   (b) $d = 100$ feet, $p = 10$ pounds, $\theta = 20^\circ$.
   (c) $d = 1{,}000$ feet, $p = 10$ pounds, $\theta = 30^\circ$.

   How far can the sled be dragged if the number of available foot pounds of work is 1,000 and if

   (d) $p = 10$ pounds, $\theta = 10^\circ$.
   (e) $p = 100$ pounds, $\theta = 20^\circ$.
   (f) $p = 100$ pounds, $\theta = 0^\circ$.
   (g) $p = 100$ pounds, $\theta = 89^\circ$.

2. The drawing shows a smooth incline d feet long which makes an angle θ with the horizontal.



   How much work is done in moving an object weighing p pounds from R to S if

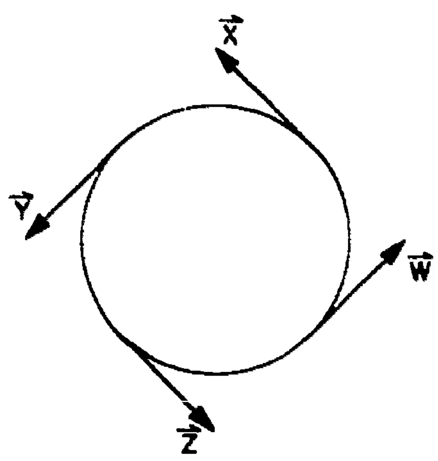
   (a) $d = 10$ feet, $p = 10$ pounds, $\theta = 10^\circ$.
   (b) $d = 100$ feet, $p = 10$ pounds, $\theta = 20^\circ$.
   (c) $d = 100$ feet, $p = 10$ pounds, $\theta = 30^\circ$.

How far can the weight be moved if the number of available foot pounds is 1,000 and if

(d) $p = 10$ pounds, $\theta = 10^\circ$.
(e) $p = 10$ pounds, $\theta = 20^\circ$.
(f) $p = 100$ pounds, $\theta = 1^\circ$.
(g) $p = 100$ pounds, $\theta = 89^\circ$.

<u>Velocity</u> is another concept of physics that can be represented by means of vectors. In ordinary language the words "speed" and "velocity" are used as synonyms. In physics the word "speed" refers to the actual time rate of change of distance (the kind of information supplied by an automobile speedometer), and "velocity" refers to the vector whose direction is the direction of the motion and whose length represents the speed on some given scale. When velocities are represented by vectors, the lengths of these vectors give the corresponding speeds.

Figure 5b shows vectors which represent some of the velocities of a body moving around a circle with constant speed.



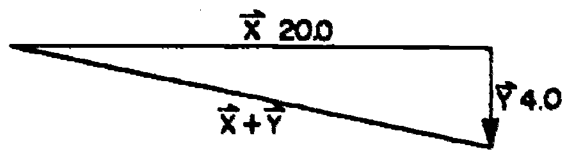
$|\vec{X}| = |\vec{Y}| = |\vec{Z}| = |\vec{W}|$

Figure 5b

It is easy to imagine situations in which velocities are compounded out of other velocities. For instance, a man walking across the deck of a moving boat has a velocity relative to the water which is compounded out of his velocity relative to the boat and out of the boat's velocity relative to the water. It is a principle of physics that the vector which represents such a compound velocity is the sum of the vectors which represent the individual velocities.

Example 5f. A ship sails east at 20.0 miles per hour. A man walks across its deck toward the south at 4.0 miles per hour. What is the man's velocity relative to the water?

Solution: In the figure, $\vec{X}$ represents the ship's velocity relative to the water, $\vec{Y}$ represents the man's velocity relative to the ship. Consequently, $\vec{X} + \vec{Y}$ represents the man's velocity relative to the water. Its length is

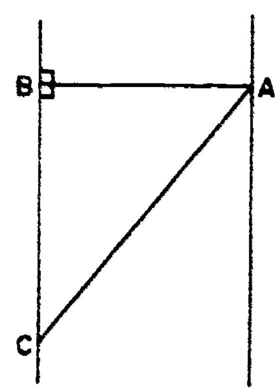


$$\sqrt{20.0^2 + 4.0^2} = 20.4$$

and its direction is 22° south of east.

Exercises 5c

1. A river 1 mile wide flows at the rate of 3 miles per hour. A man rows across the river, starting at A and aiming his boat toward B, the nearest point on the opposite shore as shown in the diagram. If it took 30 minutes for him to make the trip, how far did he row?



2. A river is $\frac{1}{2}$ mile wide and flows at the rate of 4 miles per hour. A man rows across the river in 25 minutes, landing 1.3 miles farther downstream on the opposite shore. How far did he row? In what direction did he head?

3. A river one mile wide flows at a rate of 4 miles per hour. A man wishes to row in a straight line to a point on the opposite shore two miles upstream. How fast must he row in order to make the trip in one hour?

4. A body starts at (0,0) at the time $t = 0$. It moves with constant velocity, and 20 seconds later it is at the point (40,30). Find its speed and its velocity, if one unit of length of vector corresponds to 100 feet per second.

5. A body moves with constant velocity which is represented by the vector $\vec{V} = 10\vec{i} + 10\vec{j}$. If the body is at the point (0,1) at time $t = 2$, where will it be when $t = 15$? The scale is: One unit of length of vector corresponds to 10 miles per hour; the time t is measured in hours.

6. Ship A starts from point (2,4) at time $t = 0$. Its velocity is constant, and represented by the vector $\vec{V}_a = 4\vec{i} - 3\vec{j}$. Ship B starts at the point (-1,-1) at time $t = 1$. Its velocity is also constant, and is represented by the vector $\vec{V}_b = 7\vec{i} + \vec{j}$. Will the ships collide? (Assume that a consistent scale has been used in setting up the vector representation.)

7. Ship A starts at point (2,7) at time $t = 0$. Its (constant) velocity is represented by the vector $\vec{V}_a = 3\vec{i} - 2\vec{j}$. Ship B starts at point (-1,-1) at time $t = 1$. Its (constant) velocity is represented by the vector $\vec{V}_b = 5\vec{i}$. Will the ships collide?

*8. A river is $\frac{1}{2}$ mile wide and flows at the rate of 4 miles per hour. A man can row at the rate of 3 miles per hour. If he starts from point A and rows to the opposite shore, what is the farthest point upstream at which he can reach the opposite shore? In what direction should he head?

Exercises 5d

1. Show each of the following graphically:

   (a) $3\vec{i} + 8\vec{j} + 5\vec{k}$.
   (b) $3\vec{j} + 3\vec{k}$.
   (c) $4\vec{i} + 4\vec{j}$.
   (d) $5\vec{i} + \vec{j}$.
   (e) $5\vec{i} + 5\vec{j} + \vec{k}$.
   (f) $2\vec{i} - 2\vec{j}$.
   (g) $7\vec{k}$.
   (h) $5\vec{j}$.
   (i) $7\vec{i}$.
   (j) $8\vec{i} + 8\vec{j} + 3\vec{k}$.

2. Find $\vec{A} \cdot \vec{B}$, if:

   (a) $\vec{A} = 3\vec{i} + 2\vec{j} + 4\vec{k}$; $\vec{B} = 2\vec{i} + \vec{j} + 2\vec{k}$.
   (b) $\vec{A} = 3\vec{i} + 4\vec{j} - 2\vec{k}$; $\vec{B} = 2\vec{i} + 2\vec{j} + 2\vec{k}$.
   (c) $\vec{A} = 3\vec{i} + 3\vec{k}$; $\vec{B} = 4\vec{j}$.
   (d) $\vec{A} = 4\vec{i} + 4\vec{j}$; $\vec{B} = 7\vec{k}$.
   (e) $\vec{A} = 4\vec{j} + 2\vec{k}$; $\vec{B} = 5\vec{i}$.

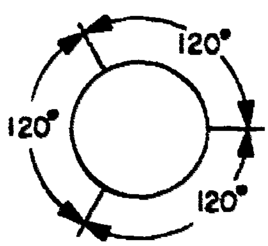
3. Find the cosine of the angle between vectors $\vec{A}$ and $\vec{B}$ in each part of Problem 2.

4. Find the cosine of the angle between the vectors $\vec{A}$ and $\vec{B}$ if $\vec{A} = 3\vec{i} + 2\vec{j} - \vec{k}$ and $\vec{B} = 4\vec{i} - 3\vec{j} + 6\vec{k}$.

*5. A lighting fixture is suspended as shown:

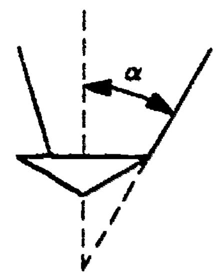
Perspective view

View from top



Vertical view

Side view (the angle is shown in its true shape; has a measure of 60° .



The fixture weighs 15 pounds. Find the tension in each of the supporting cables.

6. An airplane is climbing at an angle of 30° . Its climbing speed is 100 m.p.h. Although a wind is blowing from west to east with a velocity of 30 m.p.h., the pilot wishes to climb while heading due north. What is the ground speed of the airplane?

7. Suppose that in Problem 6 the pilot climbs at an angle of 30° , but does not insist on heading north. What is the fastest ground speed that he can achieve? Which way should he head to achieve this speed? What is the least ground speed that he can achieve? Which way should he head to achieve this?

8. Prove that

$$a(x - d) + b(y - e) + c(z - f) = 0 ,$$

is the equation of a plane through the point $Q(d,e,f)$ with the normal vector

$$\vec{N} = a\vec{i} + b\vec{j} + c\vec{k} .$$

9. Find a vector normal to the plane

$$7x - 3y + 5z = 12 .$$

10. Find the distance from the point $(0,0,0)$ to the plane

$$5x + 12y - z = 1 .$$

11. Find the distance from the plane

$$x + 2y - 3z = 1$$

to the origin.

---

6. Vectors as a Formal Mathematical System.

In our discussion of forces and velocities by means of vectors we made a few assumptions which we did not justify. We applied vector methods to the solution of force and velocity problems in a fashion which turns out to be correct but which we have not backed up with a convincing argument. Our thinking was something like this. "Some of the rules that forces obey are very much like the rules that vectors obey. Therefore we can talk abou. forces as though they were vectors." This is not really a sound argument, and if it were trusted in all cases it could lead to chaos. For instance, some of the rules that real numbers obey are the rules that integers obey, and it is not the case that real numbers can be regarded as integers.

Nevertheless, it really was correct to treat forces as vectors and we now explore a point of view which gives convincing evidence for this statement. The key fact in this examination is that every mathematical system which obeys certain of the laws which vectors obey must be essentially the same as the system of vectors itself.

We now formulate three goals:

1. To list the rules in question.
2. To give a precise specification of what we mean by saying that a mathematical system is "essentially the same" as a system of vectors.

3. To prove that systems which obey the stated rules are essentially the same as the system of vectors.

I. We state certain rules which vectors have been shown to obey. We have a set S, two operations $\oplus$, $\odot$, for which, for all α, β, γ, in S and for all real numbers r, s

(1) $\alpha \oplus \beta$ is in S.

(2) $\alpha \oplus \beta = \beta \oplus \alpha$.

(3) $\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$.

(4) There is a zero element ϕ in S such that

$$\alpha \oplus \phi = \alpha$$

(5) Each α has an additive inverse $-\alpha$ for which

$$\alpha \oplus (-\alpha) = \phi$$

(6) $r \odot \alpha$ is in S.

(7) $r \odot (s \odot \alpha) = (rs) \odot \alpha$.

(8) $(r + s) \odot \alpha = (r \odot \alpha) \oplus (s \odot \alpha)$.

(9) $r \odot (\alpha \oplus \beta) = (r \odot \alpha) \oplus (r \odot \beta)$.

(10) $1 \odot \alpha = \alpha$.

(11) There are two members β and γ of S such that each member α of S has a unique representation.

$$\alpha = (a \odot \beta) \oplus (b \odot \gamma).$$

II. We have already shown that vectors satisfy such rules, where S is interpreted as the set of vectors, $\oplus$ is interpreted as ordinary $+$ for vectors and where $\odot$ is interpreted as scalar multiplication. We take it as given (by physicists presumably) that forces also satisfy these rules, where S is the set of forces, $\alpha \oplus \beta$ means the resultant of α and β and $\odot$ means scalar multiplication. We are to show that forces are essentially the same as vectors. What do we mean by "essentially the same?" We mean that the system of forces is isomorphic to the system of vectors. What do we mean by "isomorphic"? That there is a one-to-one correspondence between the set of forces and the set of vectors such that, if force α corresponds to vector $\vec{A}$ and if force β corresponds to vector $\vec{B}$, then $\alpha \oplus \beta$ corresponds to vector $\vec{A} + \vec{B}$ and force $r \odot \alpha$ corresponds to vector $r\vec{A}$.

III. We now state and prove the promised theorem.

<u>THEOREM</u>. Any system S which satisfies Rules 1 - 11 is isomorphic to the system of vectors in a plane.

<u>Proof</u>: We first set up a one-to-one correspondence between the members of S and the vectors. For each α of S we invoke Item 11 to write

$$\alpha = (a \odot y) \oplus (b \odot w) .$$

The pair (a,b) which figures in this expression determines a unique vector $\vec{A}$, namely [a,b], which we pair with α. This process assigns to each α of S a vector $\vec{A}$ as its image. We must show that if [a,b] is the image of α and if [c,d] is the image of β, then [a + c , b + d] is the image of $\alpha + \beta$ and that [ra,rb] is the image of $r \odot \alpha$. To prove the first, write

$$\alpha = (a \odot y) \oplus (b \odot w)$$
$$\beta = (c \odot y) \oplus (d \odot w) .$$

Therefore $\alpha \oplus \beta = ((a \odot y) \oplus (b \odot w)) \oplus ((c \odot y) \oplus (d \odot w))$ which equals using Rules 2 and 3,

$$((a \odot y) \oplus (c \odot y)) \oplus ((b \odot w) \oplus (d \odot w) .$$

This in turn equals

$$((a + c) \odot y) \oplus ((b + d) \odot w)$$

by virtue of Rule 8. We see then that our one-to-one correspondence assigns [a + c , b + d] to $\alpha + \beta$.

We now examine $r \odot \alpha$. We write

$$r \odot \alpha = r \odot ((a \odot y) \oplus (b \odot w))$$

which by Rule 9 can be written as

$$r \odot (a \odot y) \oplus r \odot (b \odot w) .$$

According to Rule 7, this last equals

$$((ra) \odot y) \oplus ((rb) \odot w) ,$$

whence the image of $r \odot \alpha$ is indeed $[ra,rb]$.

This completes our proof. Notice that we did not use all the rules given. They are in fact redundant. If the last rule is left out, the remaining set of rules is not redundant, and is the set of axioms which defines a vector space. The Rules 1 - 11 are axioms for a more special mathematical system--a two-dimensional vector space.

We have shown that every system which satisfies Rules 1 - 11 is isomorphic to our system of vectors. We have not shown that the system of forces satisfies these rules. We take the physicist's word for this. We have not shown that to be "isomorphic" really means to be "essentially the same."

Exercises 6

1. Let S be the system of complex numbers. Does S satisfy Rules 1 - 11 if $\oplus$ is interpreted as ordinary addition of complex numbers and $\odot$ as ordinary multiplication of a real number by a complex number? (Hint: In checking Rule 11, try 1 for y and i for w).

2. Let S be the set of all ordered pairs (a,b) of real numbers, let $\oplus$ be defined by $(a,b) \oplus (c,d) = (a + c , b + d)$, and let $\odot$ be defined by

$$r \odot (a,b) = (\frac{ra}{2} , \frac{rb}{2}) .$$

Which of the Rules 1 - 11 does this system obey?

3. Let S be the set of all ordered pairs (a,b) of real numbers, let $\oplus$ be defined by

$$(a,b) \oplus (c,d) = \left(\frac{a+c}{2}, \frac{b+d}{2}\right),$$

and let $\odot$ be defined by

$$r \odot (a,b) = (ra,rb).$$

Which of the Rules 1 - 11 does this system obey?